DEFINITIONS

Algebra

Expressions, equations, inequalities, polynomials and the binomial expansion.

57 definitions **6** subtopics By Lucas Potrykus

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HOW TO USE THIS

Each definition gives the exact meaning of a word or symbol. Where there is a symbol, it is shown next to the name, with how to say it out loud. Most definitions have a short example.

Learn definitions word for word: in proofs and in exams, the precise wording is what earns the marks.

Searchable online, with links to every topic, at lemmamathematics.com/definitions.

EXPRESSIONS AND EQUATIONS

More in **Algebraic methods**

Variable

A letter standing for a quantity that can take different values.

Constant

A quantity whose value is fixed in the problem, written as a number or a letter (often a, b, c, k).

Term

One of the parts of an expression joined by $+$ or $-$ signs.

> EXAMPLE
> $3x^2 - 5x + 7$ has three terms.

Coefficient

The number multiplying a variable or power of a variable in a term.

> EXAMPLE
> In $3x^2 - 5x + 7$ the coefficient of x is -5; the constant term is 7.

Expression

A combination of numbers, letters and operations with no equals sign. You simplify an expression; you solve an equation.

Equation

A statement that two expressions are equal. Solving it means finding the values of the unknown that make it true.

Is identically equal to

$\equiv$

SAY *is identical to; is identically equal to*

An identity: true for **every** value of the variable, not just for some. Compare an equation such as $x^2 = 4$, which is true only for particular x. (In number theory $\equiv$ also means congruence; see below.)

> EXAMPLE
> $(x+1)^2 \equiv x^2 + 2x + 1$ and $\cos^2\theta + \sin^2\theta \equiv 1$.

More in **Trigonometric identities and equations**

Not equal to

$\neq$

SAY *is not equal to*

The two sides are different.

> EXAMPLE
> $\frac{1}{3} \neq 0.33$.

Approximately equal to

$\approx$

SAY *is approximately*

The two sides are close, usually after rounding.

> EXAMPLE
> $\pi \approx 3.142$ and $\sqrt{2} \approx 1.414$.

Is defined to be

$:=$

SAY *is defined as*

Introduces a new name for something. The new thing goes on the side with the colon. Also written $\overset{\text{def}}{=}$, or $\equiv$ in some books.

> EXAMPLE
> Let $S_n := \sum_{r=1}^{n} r$. Then $S_n = \frac{1}{2}n(n+1)$ is a theorem, not a definition.

Formula

An equation connecting quantities, used to calculate one from the others. **Rearranging** (changing the subject) makes a different quantity the one on its own.

> EXAMPLE
> $A = \pi r^2$ rearranges to $r = \sqrt{\frac{A}{\pi}}$.

Expand

Multiply out brackets.

> EXAMPLE
> $(x+2)(x-5) = x^2 - 3x - 10$.

Factorise

Write an expression as a product of factors; the reverse of expanding.

> EXAMPLE
> $x^2 - 9 = (x-3)(x+3)$, a **difference of two squares**.

Simultaneous equations

Two or more equations in two or more unknowns that must all hold at the same time. Solved by substitution or elimination; graphically, the solutions are where the graphs meet.

Is proportional to

$\propto$

SAY *is proportional to; varies as*

$y \propto x$ means $y = kx$ for some constant k.

> EXAMPLE
> If $y \propto x^2$ and $y = 12$ when $x = 2$, then $k = 3$, so $y = 3x^2$.

Inverse proportion

y is inversely proportional to x if $y = \frac{k}{x}$ for a constant k, written $y \propto \frac{1}{x}$: doubling x halves y.

QUADRATICS

More in **Algebraic methods**

Quadratic

An expression or equation of the form $ax^2 + bx + c$ with $a \neq 0$. Its graph is a parabola.

Quadratic formula

The solutions of $ax^2 + bx + c = 0$ are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Discriminant

$b^2 - 4ac$

SAY *b squared minus four a c*

$\Delta = b^2 - 4ac$ for the quadratic $ax^2 + bx + c$. Two distinct real roots if $\Delta > 0$, one repeated root if $\Delta = 0$, no real roots if $\Delta < 0$.

> EXAMPLE
> $x^2 + 2x + 5$ has $\Delta = 4 - 20 < 0$, so it never touches the x-axis.

Completing the square

Writing $ax^2 + bx + c$ as $a(x + p)^2 + q$. It shows the turning point $(-p, q)$ at a glance.

> EXAMPLE
> $x^2 + 6x + 11 = (x + 3)^2 + 2$, so the minimum point is $(-3, 2)$.

Root (zero)

A value of x that makes $f(x) = 0$. The real roots are where the graph $y = f(x)$ meets the x-axis.

Repeated root

A root that occurs more than once as a factor. The graph touches the axis there rather than crossing it (for a double root).

> EXAMPLE
> $(x - 2)^2(x + 1)$ has a repeated root at $x = 2$.

Vertex of a parabola

The turning point of a parabola: its lowest point if $a > 0$, its highest if $a < 0$. It lies on the line of symmetry $x = -\frac{b}{2a}$.

Disguised quadratic

An equation that becomes a quadratic after a substitution.

> EXAMPLE
> $x^4 - 5x^2 + 4 = 0$ with $u = x^2$ becomes $u^2 - 5u + 4 = 0$.

INDICES AND SURDS

More in **Algebraic methods**

Index (power, exponent)

In a^n, a is the base and n the index. The laws: $a^m a^n = a^{m+n}$, $\frac{a^m}{a^n} = a^{m-n}$ and $(a^m)^n = a^{mn}$.

Zero and negative indices

$a^0 = 1$ and $a^{-n} = \frac{1}{a^n}$ for $a \neq 0$, which keeps the index laws working.

> EXAMPLE
> $2^{-3} = \frac{1}{8}$.

Fractional index

$a^{\frac{1}{n}} = \sqrt[n]{a}$ and $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$.

> EXAMPLE
> $8^{\frac{2}{3}} = 2^2 = 4$.

Square root

$\sqrt{x}$

SAY *root x; the square root of x*

The **non-negative** number whose square is x. So $\sqrt{9} = 3$, not ± 3; the $\pm$ appears only when you solve an equation such as $x^2 = 9$.

> EXAMPLE
> $\sqrt{x^2} = |x|$, not x: try $x = -5$.

Plus or minus

$\pm, \mp$

SAY *plus or minus; minus or plus*

Shorthand for two cases at once. When $\pm$ and $\mp$ appear together, take the top signs together and the bottom signs together.

> EXAMPLE
> $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, and $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$.

Surd

An irrational root such as $\sqrt{2}$ or $\sqrt[3]{5}$, left in root form so the value stays exact.

> EXAMPLE
> $\sqrt{12} = 2\sqrt{3}$, but $\sqrt{9} = 3$ is not a surd.

Rationalising the denominator

Rewriting a fraction so that its denominator has no surds, by multiplying top and bottom by a suitable expression (for $a + \sqrt{b}$, the **conjugate** $a - \sqrt{b}$).

> EXAMPLE
> $$\frac{1}{3 + \sqrt{2}} = \frac{3 - \sqrt{2}}{9 - 2} = \frac{3 - \sqrt{2}}{7}.$$

Reciprocal

The reciprocal of $x \neq 0$ is $\frac{1}{x}$; a number times its reciprocal is 1.

Standard form

A number written as $A \times 10^n$ with $1 \leq A < 10$ and n an integer.

> EXAMPLE
> $0.00042 = 4.2 \times 10^{-4}$.

INEQUALITIES

More in **Algebraic methods**

Less than or equal to, greater than or equal to

$\leq$, $\geq$

SAY *is at most; is at least*

$x \leq 3$ allows $x = 3$; $x < 3$ does not. British exam papers often use the slanted forms $\leqslant$ and $\geqslant$, which mean the same.

> EXAMPLE
> $x^2 \leq 9 \iff -3 \leq x \leq 3$.

Much less than, much greater than

$\ll$, $\gg$

SAY *is much less than; is much greater than*

Used in modelling and approximation: one quantity is so much smaller that it can be ignored.

> EXAMPLE
> If $h \ll 1$ then h^2 is tiny compared with h, so $(1+h)^2 \approx 1 + 2h$.

More in **Binomial expansion**

Strict inequality

One using $<$ or $>$, so equality is not allowed. A non-strict inequality uses $\leq$ or $\geq$.

Rules for inequalities

You can add or subtract anything on both sides. Multiplying or dividing by a **negative** number reverses the inequality sign.

> EXAMPLE
> $-2x < 6 \iff x > -3$.

Critical values

The values where the two sides of an inequality are equal. For a quadratic inequality, find them first and then decide which side of each the solution lies on, usually from a sketch.

> EXAMPLE
> $x^2 - x - 6 > 0$: critical values -2 and 3, solution $x < -2$ or $x > 3$.

Region

The set of points in the plane satisfying one or more inequalities, shaded on a graph. A dashed boundary means it is not included (strict inequality); a solid one means it is.

POLYNOMIALS AND ALGEBRAIC FRACTIONS

More in **Polynomials**

Polynomial

An expression $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ in which every power of x is a non-negative integer.

> EXAMPLE
> $4x^3 - x + 2$ is a polynomial; $\sqrt{x}$ and $\frac{1}{x}$ are not.

Degree

The highest power of the variable in a polynomial with a non-zero coefficient. Linear, quadratic, cubic and quartic mean degree 1, 2, 3 and 4.

Leading coefficient

The coefficient of the highest power. A polynomial whose leading coefficient is 1 is called **monic**.

Factor theorem

$(x - a)$ is a factor of the polynomial $f(x)$ if and only if $f(a) = 0$. More generally $(ax - b)$ is a factor if and only if $f\left(\frac{b}{a}\right) = 0$.

> EXAMPLE
> $f(x) = x^3 - 7x + 6$ has $f(1) = 0$, so $(x - 1)$ is a factor.

Remainder theorem

When the polynomial $f(x)$ is divided by $(x - a)$, the remainder is $f(a)$.

Algebraic division

Dividing one polynomial by another to give a quotient and a remainder: $f(x) = (\text{divisor})(\text{quotient}) + \text{remainder}$, where the remainder has lower degree than the divisor.

Algebraic fraction

A fraction whose numerator and denominator are algebraic expressions. Simplify by factorising and cancelling common **factors** (never terms).

Proper and improper algebraic fractions

A fraction of polynomials is proper if the degree of the numerator is less than that of the denominator, and improper otherwise. An improper one can be divided out first.

> EXAMPLE
> $\frac{x^2+1}{x-1}$ is improper; it equals $x + 1 + \frac{2}{x-1}$.

More in **Partial fractions**

Partial fractions

Splitting a fraction with a factorised denominator into a sum of simpler fractions, one for each factor. Used for integration and for series expansions.

> EXAMPLE
> $$\frac{5}{(x-2)(x+3)} = \frac{1}{x-2} - \frac{1}{x+3}.$$

More in **Partial fractions**

Roots and coefficients

For a polynomial with roots $\alpha, \beta, \ldots$, the coefficients give the sums and products of the roots. For $ax^2 + bx + c$: $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$.

> EXAMPLE
> For $ax^3 + bx^2 + cx + d$: $\sum \alpha = -\frac{b}{a}$, $\sum \alpha\beta = \frac{c}{a}$, $\alpha\beta\gamma = -\frac{d}{a}$.

More in **Roots of polynomials and series**

COUNTING AND THE BINOMIAL EXPANSION

More in **Binomial expansion**

Factorial

$n!$

SAY *n factorial*

$n! = n \times (n-1) \times \cdots \times 2 \times 1$, the number of ways to arrange n different objects in a row. By definition $0! = 1$ (there is one way to arrange nothing).

> EXAMPLE
> $5! = 120$.

Binomial coefficient

$\binom{n}{r}$

SAY *n choose r*

The number of ways to choose r objects from n when order does not matter: $\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$. Also written ${}^nC_r$ or ${}_nC_r$.

> EXAMPLE
> $\binom{5}{2} = 10$, so $(1+x)^5$ has $10x^2$ as its x^2 term.

Permutations

${}^nP_r$

SAY *n P r*

The number of ways to choose r objects from n **in order**: $\frac{n!}{(n-r)!}$.

> EXAMPLE
> ${}^5P_2 = 20$: first and second place from 5 runners.

More in **Probability**

Permutation

An arrangement in which order matters. There are $n!$ ways to arrange n different objects in a row.

> EXAMPLE
> The letters of CAT can be arranged in $3! = 6$ ways.

Combination

A selection in which order does not matter. There are $\binom{n}{r}$ ways to choose r objects from n.

> EXAMPLE
> A team of 3 from 10 people: $\binom{10}{3} = 120$.

Pascal's triangle

The triangle of numbers in which each entry is the sum of the two above it. Row n (starting from row 0) lists $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$.

> EXAMPLE
> Row 4 is $1, 4, 6, 4, 1$.

Binomial expansion

For a positive integer n: $(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$. For other n, $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots$ is an infinite series valid when $|x| < 1$.

> EXAMPLE
> $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$ for $|x| < 1$.

Range of validity

The values of x for which an infinite series expansion converges to the function. For $(a+bx)^n$ with n not a positive integer, it is $\left|\frac{bx}{a}\right| < 1$.

Spotted a mistake, or a definition that should be here? Email lpotry31@gmail.com. More at lemmamathematics.com.