REFERENCE

Definitions

Every word and symbol in school mathematics, explained: from the first ideas of algebra and proof to the end of Further Maths.

567 definitions **16** topics **62** subtopics By Lucas Potrykus

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HOW TO USE THIS

Each definition gives the exact meaning of a word or symbol. Where there is a symbol, it is shown next to the name, with how to say it out loud. Most definitions have a short example.

Learn definitions word for word: in proofs and in exams, the precise wording is what earns the marks.

Searchable online, with links to every topic, at lemmamathematics.com/definitions.

Logic and proof

The language every proof is written in: statements, quantifiers and the standard methods of proof.

STATEMENTS AND CONNECTIVES

More in **Proof I**

Statement (proposition)

A sentence that is either true or false, but not both. Questions, commands and expressions such as $x^2 + 1$ are not statements; "$x^2 + 1 > 0$ for every real x" is.

EXAMPLE

"7 is prime" is a true statement; "every odd number is prime" is a false one.

Implies

$\Rightarrow$

SAY *implies; if … then …*

$P \Rightarrow Q$ means that whenever P is true, Q is true as well. It makes no claim about what happens when P is false.

> EXAMPLE
> $x = 2 \Rightarrow x^2 = 4$. The arrow does not reverse: $x^2 = 4$ also allows $x = -2$.

Is implied by

$\Leftarrow$

SAY *is implied by; if*

$P \Leftarrow Q$ means the same as $Q \Rightarrow P$. Useful when you want to keep the conclusion on the left.

> EXAMPLE
> $x^2 = 4 \Leftarrow x = 2$ is true.

If and only if

$\iff$

SAY *if and only if; "iff"*

$P \iff Q$ means $P \Rightarrow Q$ **and** $Q \Rightarrow P$: the two statements are true in exactly the same situations. To prove one, you usually prove both directions separately.

> EXAMPLE
> For real x: $x^3 = 8 \iff x = 2$.

Not

$\neg$

SAY *not*

The negation of a statement: $\neg P$ is true exactly when P is false. Also written $\sim P$ or $\overline{P}$. Negating swaps the quantifiers: $\neg(\forall x,\ P(x))$ is the same as $\exists x,\ \neg P(x)$.

> EXAMPLE
> The negation of "every prime is odd" is "some prime is even", and 2 shows that the negation is true.

And

$\wedge$

SAY *and*

$P \wedge Q$ is true only when both P and Q are true.

> EXAMPLE
> $x > 0 \wedge x < 1$ says the same as $0 < x < 1$.

Or

$\vee$

SAY *or*

$P \vee Q$ is true when at least one of P, Q is true. In mathematics "or" is always **inclusive**: both being true is allowed.

> EXAMPLE
> $x^2 = 1 \iff x = 1 \vee x = -1$.

Converse

The converse of $P \Rightarrow Q$ is $Q \Rightarrow P$. A statement and its converse are logically independent: one can be true while the other is false.

> EXAMPLE
> "If n is divisible by 4 then n is even" is true; its converse, "if n is even then n is divisible by 4", is false ($n = 6$).

Contrapositive

The contrapositive of $P \Rightarrow Q$ is $\neg Q \Rightarrow \neg P$. It is always **equivalent** to the original statement, so proving the contrapositive proves the statement.

> EXAMPLE
> "If n^2 is even then n is even" has contrapositive "if n is odd then n^2 is odd", which is easy to prove directly.

Inverse of a statement

The inverse of $P \Rightarrow Q$ is $\neg P \Rightarrow \neg Q$. It is equivalent to the converse, not to the original statement.

Necessary condition

Q is necessary for P when P cannot happen without Q, that is $P \Rightarrow Q$.

> EXAMPLE
> Being even is necessary for being divisible by 4, but it is not sufficient.

Sufficient condition

P is sufficient for Q when P on its own guarantees Q, that is $P \Rightarrow Q$. "Necessary and sufficient" means $P \iff Q$.

> EXAMPLE
> $x = 3$ is sufficient for $x^2 = 9$ but not necessary, since $x = -3$ also works.

Tautology

A compound statement that is true whatever the truth values of its parts. Its opposite, which is always false, is a contradiction.

> EXAMPLE
> $P \vee \neg P$ is a tautology; $P \wedge \neg P$ is a contradiction.

Truth table

A table listing every combination of truth values of the basic statements, and the resulting truth value of a compound statement. With n basic statements there are 2^n rows.

Therefore $\therefore$

SAY *therefore; so*

Introduces a conclusion that follows from what came before. In a written proof, words such as "so" or "hence" are usually clearer.

> EXAMPLE
> $2x + 1 = 7, \therefore x = 3$.

Because $\because$

SAY *because; since*

Introduces a reason. Rarely used in modern writing, where "since" reads better.

> EXAMPLE
> $\sqrt{x^2} = x \because x \geq 0$.

QUANTIFIERS

More in **Proof I**

For all

$\forall$

SAY *for all; for every*

The universal quantifier. $\forall x \in S,\ P(x)$ says that $P(x)$ is true for every element x of S. One counterexample is enough to make it false.

> EXAMPLE
> $\forall x \in \mathbb{R},\ x^2 \geq 0$: every real number has a non-negative square.

There exists

$\exists$

SAY *there exists; there is at least one*

The existential quantifier. $\exists x \in S$ such that $P(x)$ says that at least one element of S makes $P(x)$ true. One example is enough to prove it.

> EXAMPLE
> $\exists x \in \mathbb{R}$ such that $x^2 = 2$ (namely $x = \sqrt{2}$). But there is no such $x \in \mathbb{Q}$.

There exists a unique

$\exists!$

SAY *there exists exactly one*

There is one, and only one, element with the property. Proving it takes two parts: existence (find one) and uniqueness (show any two must be equal).

> EXAMPLE
> $\exists!\, x \in \mathbb{R}$ such that $x^3 = 8$. But not $\exists!\, x$ with $x^2 = 4$, since there are two.

Counterexample

A single example showing that a "for all" statement is false. One counterexample is a complete disproof; no number of examples is a proof.

> EXAMPLE
> $n = 41$ is a counterexample to "$n^2 + n + 41$ is prime for every natural number n", since $41^2 + 41 + 41 = 41 \times 43$.

Arbitrary

"Let x be arbitrary" means x stands for any element at all, with no special properties assumed. Whatever is then proved about x holds for every element, which is how "for all" statements are proved.

THEOREMS AND THE WORDS AROUND THEM

More in **Proof I**

Axiom

A statement accepted as true without proof, used as a starting point. Every proof ultimately rests on axioms and definitions.

> EXAMPLE
> Euclid's axiom that a straight line can be drawn between any two points.

Definition

A precise statement of what a word or symbol means. Definitions are not proved; they are agreed.

Theorem

An important statement that has been proved.

> EXAMPLE
> Pythagoras' theorem; the fundamental theorem of calculus.

Lemma

A smaller result proved mainly as a step towards a theorem. (This site is named after it.)

Corollary

A result that follows quickly from a theorem just proved.

> EXAMPLE
> From "the angles of a triangle add to $180°$" follows the corollary "each angle of an equilateral triangle is $60°$".

Conjecture

A statement believed to be true but not yet proved.

> EXAMPLE
> Goldbach's conjecture: every even number greater than 2 is the sum of two primes.

Proof

A logical argument that establishes, beyond any doubt, that a statement is true, starting from definitions, axioms and results already proved.

Without loss of generality

"WLOG" announces that I am about to assume something special (such as $a \leq b$) because the other cases work in exactly the same way.

> EXAMPLE
> For two integers, WLOG $a \leq b$: otherwise swap their names.

Hence and hence or otherwise

In an exam, "hence" means you must use the previous result; "hence or otherwise" means using it is the quickest route, but any correct method earns the marks.

Show that

An instruction to reach a given answer, with every step written out. Because the answer is printed, the marks are all for the working.

End of proof

■

SAY *QED; "which was to be shown"*

Marks the end of a proof. Also written □ or QED, from the Latin *quod erat demonstrandum*, "which was to be shown".

METHODS OF PROOF

More in **Proof I**

Proof by deduction (direct proof)

Start from known facts or the hypothesis and reach the conclusion by a chain of logical steps.

> EXAMPLE
> If $m = 2a$ and $n = 2b$ are even then $m + n = 2(a + b)$ is even.

Proof by exhaustion

Split the statement into a finite number of cases and check every one.

> EXAMPLE
> Every square number leaves remainder 0 or 1 on division by 3: check $n = 3k$, $3k + 1$ and $3k + 2$.

Proof by contradiction

Assume the statement is false, and deduce something impossible. The assumption must therefore be wrong, so the statement is true.

> EXAMPLE
> The classic proofs that $\sqrt{2}$ is irrational and that there are infinitely many primes.

Contradiction

$\Rightarrow\!\Leftarrow$

SAY *contradiction*

Marks the moment in a proof by contradiction where two statements clash, so the original assumption must be false. Also written $\bot$, # or simply the word "contradiction".

> EXAMPLE
> Assume $\sqrt{2} = \frac{p}{q}$ in lowest terms … so p and q are both even $\Rightarrow\!\Leftarrow$.

Proof by contrapositive

To prove $P \Rightarrow Q$, prove the equivalent statement $\neg Q \Rightarrow \neg P$ instead.

Disproof by counterexample

Showing a "for all" statement is false by giving one case where it fails.

Proof by induction

Prove a statement for a first value (the base case), then prove that if it holds for $n = k$ it holds for $n = k + 1$. Together these prove it for every integer from the first value onwards.

More in **Mathematical induction**

INTEGERS, PRIMES AND DIVISIBILITY

More in **Proof I**

Integer

A whole number: positive, negative or zero. The set of integers is $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

Even and odd

An integer is even if it can be written $2k$ for an integer k, and odd if it can be written $2k + 1$. Every integer is exactly one of the two; this is its **parity**.

Factor (divisor)

a is a factor of b if $b = ak$ for some integer k.

> EXAMPLE
> The factors of 12 are 1, 2, 3, 4, 6 and 12.

Multiple

b is a multiple of a if $b = ak$ for some integer k.

> EXAMPLE
> The multiples of 7 are $\ldots, -7, 0, 7, 14, \ldots$

Divides

$a \mid b$

SAY *a divides b*

a divides b exactly: $b = ka$ for some integer k. Not to be confused with $\frac{a}{b}$, which is a number; $a \mid b$ is a statement that is true or false.

> EXAMPLE
> $3 \mid 12$ but $5 \nmid 12$.

Prime number

A natural number greater than 1 whose only positive factors are 1 and itself. 1 is **not** prime.

> EXAMPLE
> 2, 3, 5, 7, 11, 13, … 2 is the only even prime.

Composite number

A natural number greater than 1 that is not prime, so it has a factor other than 1 and itself.

> EXAMPLE
> $91 = 7 \times 13$ is composite.

Fundamental theorem of arithmetic

Every integer greater than 1 can be written as a product of primes, and this **prime factorisation** is unique apart from the order of the factors.

> EXAMPLE
> $360 = 2^3 \times 3^2 \times 5$.

Greatest common divisor

$\gcd(a, b)$

SAY *g c d of a and b; highest common factor*

The largest integer dividing both a and b, also written $\operatorname{hcf}(a, b)$ or just (a, b). The partner is $\operatorname{lcm}(a, b)$, the lowest common multiple, and $\gcd(a, b) \cdot \operatorname{lcm}(a, b) = ab$ for positive integers.

> EXAMPLE
> $\gcd(12, 18) = 6$ and $\operatorname{lcm}(4, 6) = 12$.

Coprime

Two integers are coprime when their only common positive factor is 1, that is $\gcd(a, b) = 1$.

> EXAMPLE
> 8 and 15 are coprime, although neither is prime.

Quotient and remainder

For integers a and $b > 0$ there are unique integers q and r with $a = bq + r$ and $0 \le r < b$. q is the quotient and r the remainder.

> EXAMPLE
> $47 = 6 \times 7 + 5$: quotient 7, remainder 5.

Congruent modulo n

$a \equiv b \pmod{n}$

SAY *a is congruent to b mod n*

a and b leave the same remainder when divided by n; equivalently $n \mid a - b$. It is the arithmetic of a clock with n hours.

> EXAMPLE
> $17 \equiv 2 \pmod{5}$, and $10^k \equiv 1 \pmod{9}$, which is why the digit-sum test for 9 works.

Rational number

A number that can be written $\frac{p}{q}$ with p, q integers and $q \neq 0$. Their decimals either stop or repeat.

> EXAMPLE
> $0.\overline{3} = \frac{1}{3}$ and $-2.5 = -\frac{5}{2}$ are rational.

Irrational number

A real number that is not rational. Its decimal expansion never ends and never repeats.

> EXAMPLE
> $\sqrt{2}$, π and e are irrational.

Perfect square

An integer that is the square of an integer. Similarly a perfect cube is the cube of an integer.

> EXAMPLE
> $0, 1, 4, 9, 16, \ldots$

Sets and number systems

Collections of objects, the standard sets of numbers, and intervals of the number line.

SETS

Set

A collection of distinct objects, called its elements or members. A set is determined entirely by which elements it contains.

Set

$\{\,\}$

SAY *the set containing …*

Curly brackets list the elements of a set. Order and repetition do not matter, so $\{1, 2\} = \{2, 1\} = \{1, 1, 2\}$.

> EXAMPLE
> $\{2, 4, 6, 8\}$ is the set of even numbers from 2 to 8.

Set-builder notation

$\{x : P(x)\}$

SAY *the set of all x such that $P(x)$*

Describes a set by a property its elements share. The colon is read "such that"; a vertical bar, $\{x \mid P(x)\}$, means the same.

> EXAMPLE
> $\{x \in \mathbb{R} : x^2 < 4\}$ is the interval $(-2, 2)$.

Is an element of

$\in$

SAY *is in; is an element of; belongs to*

$x \in A$ says that x is one of the objects in the set A.

> EXAMPLE
> $3 \in \{1, 2, 3\}$ and $\sqrt{2} \in \mathbb{R}$.

Is not an element of

$\notin$

SAY *is not in*

$x \notin A$ says that x is not in the set A.

> EXAMPLE
> $\sqrt{2} \notin \mathbb{Q}$: the square root of 2 is irrational.

More in **Proof I**

Empty set

$\varnothing$

SAY *the empty set*

The set with no elements, also written $\{\}$. Careful: $\{\varnothing\}$ is not empty, because it has one element (the empty set itself).

> EXAMPLE
> $\{x \in \mathbb{R} : x^2 = -1\} = \varnothing$.

Universal set

ξ

SAY *"ksy" or "zai"*

The set of everything under consideration, drawn as the rectangle in a Venn diagram. Also written $\mathcal{E}$ or U.

> EXAMPLE
> With $\xi = \{1, 2, 3, 4, 5, 6\}$ for a dice roll, the event "even" is $\{2, 4, 6\}$.

More in **Probability**

Equal sets

Two sets are equal when they have exactly the same elements. To prove $A = B$, show $A \subseteq B$ and $B \subseteq A$.

Is a subset of

$\subseteq$

SAY *is a subset of; is contained in*

$A \subseteq B$ means every element of A is also in B. Every set is a subset of itself, and $\varnothing$ is a subset of every set.

> EXAMPLE
> $\{1, 2\} \subseteq \{1, 2, 3\}$ and $\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$.

Is a proper subset of

$\subset$

SAY *is a proper subset of*

$A \subset B$ usually means $A \subseteq B$ and $A \neq B$. Some books use $\subset$ for any subset and write $\subsetneq$ for a proper one, so check the convention.

> EXAMPLE
> $\{1, 2\} \subset \{1, 2, 3\}$, but $\{1, 2, 3\} \not\subset \{1, 2, 3\}$.

Finite and infinite sets

A set is finite if it has a whole number of elements, and infinite otherwise.

> EXAMPLE
> $\{x \in \mathbb{Z} : x^2 < 10\}$ is finite (7 elements); the set of primes is infinite.

Number of elements

$|A|$

SAY *the size of A; the cardinality of A*

How many elements A has. At school this is often written $n(A)$. For infinite sets, comparing sizes leads to the surprising world of cardinals (see $\aleph_0$ below).

> EXAMPLE
> $|\{1, 2, 3\}| = 3$ and $|\varnothing| = 0$.

Power set

$\mathcal{P}(A)$

SAY *the power set of A*

The set of all subsets of A. If $|A| = n$ then $|\mathcal{P}(A)| = 2^n$, since each element is either in a subset or not.

> EXAMPLE
> $\mathcal{P}(\{1, 2\}) = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$.

OPERATIONS ON SETS

More in **Probability**

Union

$\cup$

SAY *union; "or"*

$A \cup B$ is the set of elements in A, in B, or in both.

> EXAMPLE
> $\{1, 2, 3\} \cup \{2, 3, 4\} = \{1, 2, 3, 4\}$.

Intersection

$\cap$

SAY *intersection; "and"*

$A \cap B$ is the set of elements in both A and B. If $A \cap B = \varnothing$ the sets are called disjoint (in probability, mutually exclusive).

> EXAMPLE
> $\{1, 2, 3\} \cap \{2, 3, 4\} = \{2, 3\}$.

Complement

A'

SAY *A dash; not A; A complement*

Everything in the universal set that is not in A. Also written A^c or $\overline{A}$.

> EXAMPLE
> With $\xi = \{1, \ldots, 6\}$ and $A = \{1, 2\}$, $A' = \{3, 4, 5, 6\}$. In probability, $\mathrm{P}(A') = 1 - \mathrm{P}(A)$.

Set difference

$A \setminus B$

SAY *A minus B; A without B*

The elements of A that are not in B. Also written $A - B$. It equals $A \cap B'$.

> EXAMPLE
> $\{1, 2, 3\} \setminus \{2, 3, 4\} = \{1\}$, and $\mathbb{R} \setminus \{0\}$ is the set of non-zero reals.

Disjoint sets

Sets with no elements in common: $A \cap B = \emptyset$.

> EXAMPLE
> The even and the odd integers are disjoint.

Venn diagram

A diagram showing sets as overlapping regions inside a rectangle that represents the universal set.

Cartesian product

$A \times B$

SAY *A cross B*

The set of all ordered pairs (a, b) with $a \in A$ and $b \in B$. If A has m elements and B has n, then $A \times B$ has mn.

> EXAMPLE
> $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ is the coordinate plane; $\{0, 1\} \times \{x, y\} = \{(0, x), (0, y), (1, x), (1, y)\}$.

Natural numbers

$\mathbb{N}$

SAY *N; the naturals*

The counting numbers. Most school books take $\mathbb{N} = \{1, 2, 3, \ldots\}$, but many university courses include 0. If it matters, $\mathbb{Z}^+$ and $\mathbb{Z}_{\geq 0}$ are unambiguous.

> EXAMPLE
> $5 \in \mathbb{N}$ but $-5 \notin \mathbb{N}$.

Integers

$\mathbb{Z}$

SAY *Z; the integers*

All whole numbers, positive, negative and zero: $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$. The Z is from the German *Zahlen*, "numbers". $\mathbb{Z}^+$ means the positive integers.

> EXAMPLE
> $-7 \in \mathbb{Z}$ but $\frac{1}{2} \notin \mathbb{Z}$.

Rational numbers

$\mathbb{Q}$

SAY *Q; the rationals*

Numbers that can be written as $\frac{p}{q}$ with $p, q \in \mathbb{Z}$ and $q \neq 0$. Q is for "quotient". Their decimals terminate or repeat.

> EXAMPLE
> $0.75 = \frac{3}{4} \in \mathbb{Q}$ and $0.\dot{3} = \frac{1}{3} \in \mathbb{Q}$, but $\pi \notin \mathbb{Q}$.

More in **Proof I**

Real numbers

$\mathbb{R}$

SAY *R; the reals*

Every point on the number line, rational or irrational. $\mathbb{R}^+$ means the positive reals, and $\mathbb{R}^n$ is the set of lists of n real numbers, such as points in space when $n = 3$.

> EXAMPLE
> $\sqrt{2}, \pi, \mathrm{e}, -4 \in \mathbb{R}$, but $\mathrm{i} \notin \mathbb{R}$.

Complex numbers

$\mathbb{C}$

SAY *C; the complex numbers*

Numbers $a + b\mathrm{i}$ with $a, b \in \mathbb{R}$ and $\mathrm{i}^2 = -1$. Every polynomial equation has all its roots in $\mathbb{C}$.

> EXAMPLE
> $3 - 2\mathrm{i} \in \mathbb{C}$, and $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$.

More in **Complex numbers**

Positive, negative and non-negative

Positive means > 0 and negative means < 0. Non-negative means ≥ 0, so zero is non-negative but not positive.

Number line

A line on which every real number has its own point, increasing from left to right. The real numbers fill it with no gaps.

Pi

π

SAY *pie*

The ratio of a circle's circumference to its diameter, $\pi \approx 3.14159$. It is irrational, and even transcendental (it is not the root of any polynomial with integer coefficients).

> EXAMPLE
> A full turn is 2π radians.

More in **Trigonometry**

Infinity

∞

SAY *infinity*

Not a real number, but a shorthand for "grows without bound". So expressions like $\infty - \infty$ have no fixed value, and you cannot cancel infinities.

> EXAMPLE
> $\sum_{r=1}^{\infty} \frac{1}{2^r} = 1$ means the partial sums get as close to 1 as you like.

More in **Sequences and series**

INTERVALS AND BOUNDS

More in **Functions and graphs**

Closed interval

$[a, b]$

SAY *the closed interval from a to b*

All real x with $a \le x \le b$: square brackets include the endpoint.

> EXAMPLE
> $[0, 1]$ contains 0, 1 and everything in between.

Open interval

(a, b)

SAY *the open interval from a to b*

All real x with $a < x < b$: round brackets exclude the endpoint. Also written $]a, b[$ in some countries. Infinity always gets a round bracket, as in $[0, \infty)$. Context tells you whether (a, b) is an interval or a point.

> EXAMPLE
> $x \in (2, 5]$ means $2 < x \le 5$.

Half-open interval

An interval containing one endpoint but not the other: $[a, b) = \{x : a \le x < b\}$ and $(a, b] = \{x : a < x \le b\}$.

Bounded

A set of numbers is bounded above if some number M is at least as large as every element (an upper bound), bounded below if some number is no larger than every element, and bounded if both.

> EXAMPLE
> $(0, 1)$ is bounded; $\mathbb{N}$ is bounded below but not above.

Algebra

Expressions, equations, inequalities, polynomials and the binomial expansion.

EXPRESSIONS AND EQUATIONS

More in **Algebraic methods**

Variable

A letter standing for a quantity that can take different values.

Constant

A quantity whose value is fixed in the problem, written as a number or a letter (often a, b, c, k).

Term

One of the parts of an expression joined by $+$ or $-$ signs.

> EXAMPLE
> $3x^2 - 5x + 7$ has three terms.

Coefficient

The number multiplying a variable or power of a variable in a term.

> EXAMPLE
> In $3x^2 - 5x + 7$ the coefficient of x is -5; the constant term is 7.

Expression

A combination of numbers, letters and operations with no equals sign. You simplify an expression; you solve an equation.

Equation

A statement that two expressions are equal. Solving it means finding the values of the unknown that make it true.

Is identically equal to

$\equiv$

SAY *is identical to; is identically equal to*

An identity: true for **every** value of the variable, not just for some. Compare an equation such as $x^2 = 4$, which is true only for particular x. (In number theory $\equiv$ also means congruence; see below.)

> EXAMPLE
> $(x+1)^2 \equiv x^2 + 2x + 1$ and $\cos^2\theta + \sin^2\theta \equiv 1$.

More in **Trigonometric identities and equations**

Not equal to

$\neq$

SAY *is not equal to*

The two sides are different.

> EXAMPLE
> $\frac{1}{3} \neq 0.33$.

Approximately equal to

$\approx$

SAY *is approximately*

The two sides are close, usually after rounding.

> EXAMPLE
> $\pi \approx 3.142$ and $\sqrt{2} \approx 1.414$.

Is defined to be

$:=$

SAY *is defined as*

Introduces a new name for something. The new thing goes on the side with the colon. Also written $\overset{\text{def}}{=}$, or $\equiv$ in some books.

> EXAMPLE
> Let $S_n := \sum_{r=1}^{n} r$. Then $S_n = \frac{1}{2}n(n+1)$ is a theorem, not a definition.

Formula

An equation connecting quantities, used to calculate one from the others. **Rearranging** (changing the subject) makes a different quantity the one on its own.

> EXAMPLE
> $A = \pi r^2$ rearranges to $r = \sqrt{\frac{A}{\pi}}$.

Expand

Multiply out brackets.

> EXAMPLE
> $(x+2)(x-5) = x^2 - 3x - 10$.

Factorise

Write an expression as a product of factors; the reverse of expanding.

> EXAMPLE
> $x^2 - 9 = (x-3)(x+3)$, a **difference of two squares**.

Simultaneous equations

Two or more equations in two or more unknowns that must all hold at the same time. Solved by substitution or elimination; graphically, the solutions are where the graphs meet.

Is proportional to

$\propto$

SAY *is proportional to; varies as*

$y \propto x$ means $y = kx$ for some constant k.

> EXAMPLE
> If $y \propto x^2$ and $y = 12$ when $x = 2$, then $k = 3$, so $y = 3x^2$.

Inverse proportion

y is inversely proportional to x if $y = \frac{k}{x}$ for a constant k, written $y \propto \frac{1}{x}$: doubling x halves y.

QUADRATICS

More in **Algebraic methods**

Quadratic

An expression or equation of the form $ax^2 + bx + c$ with $a \neq 0$. Its graph is a parabola.

Quadratic formula

The solutions of $ax^2 + bx + c = 0$ are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Discriminant

$b^2 - 4ac$

SAY *b squared minus four a c*

$\Delta = b^2 - 4ac$ for the quadratic $ax^2 + bx + c$. Two distinct real roots if $\Delta > 0$, one repeated root if $\Delta = 0$, no real roots if $\Delta < 0$.

> EXAMPLE
> $x^2 + 2x + 5$ has $\Delta = 4 - 20 < 0$, so it never touches the x-axis.

Completing the square

Writing $ax^2 + bx + c$ as $a(x + p)^2 + q$. It shows the turning point $(-p, q)$ at a glance.

> EXAMPLE
> $x^2 + 6x + 11 = (x + 3)^2 + 2$, so the minimum point is $(-3, 2)$.

Root (zero)

A value of x that makes $f(x) = 0$. The real roots are where the graph $y = f(x)$ meets the x-axis.

Repeated root

A root that occurs more than once as a factor. The graph touches the axis there rather than crossing it (for a double root).

> EXAMPLE
> $(x - 2)^2(x + 1)$ has a repeated root at $x = 2$.

Vertex of a parabola

The turning point of a parabola: its lowest point if $a > 0$, its highest if $a < 0$. It lies on the line of symmetry $x = -\frac{b}{2a}$.

Disguised quadratic

An equation that becomes a quadratic after a substitution.

> EXAMPLE
> $x^4 - 5x^2 + 4 = 0$ with $u = x^2$ becomes $u^2 - 5u + 4 = 0$.

INDICES AND SURDS

More in **Algebraic methods**

Index (power, exponent)

In a^n, a is the base and n the index. The laws: $a^m a^n = a^{m+n}$, $\frac{a^m}{a^n} = a^{m-n}$ and $(a^m)^n = a^{mn}$.

Zero and negative indices

$a^0 = 1$ and $a^{-n} = \frac{1}{a^n}$ for $a \neq 0$, which keeps the index laws working.

> EXAMPLE
> $2^{-3} = \frac{1}{8}$.

Fractional index

$a^{\frac{1}{n}} = \sqrt[n]{a}$ and $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$.

> EXAMPLE
> $8^{\frac{2}{3}} = 2^2 = 4$.

Square root

$\sqrt{x}$

SAY *root x; the square root of x*

The **non-negative** number whose square is x. So $\sqrt{9} = 3$, not ± 3; the $\pm$ appears only when you solve an equation such as $x^2 = 9$.

> EXAMPLE
> $\sqrt{x^2} = |x|$, not x: try $x = -5$.

Plus or minus

$\pm, \mp$

SAY *plus or minus; minus or plus*

Shorthand for two cases at once. When $\pm$ and $\mp$ appear together, take the top signs together and the bottom signs together.

> EXAMPLE
> $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, and $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$.

Surd

An irrational root such as $\sqrt{2}$ or $\sqrt[3]{5}$, left in root form so the value stays exact.

> EXAMPLE
> $\sqrt{12} = 2\sqrt{3}$, but $\sqrt{9} = 3$ is not a surd.

Rationalising the denominator

Rewriting a fraction so that its denominator has no surds, by multiplying top and bottom by a suitable expression (for $a + \sqrt{b}$, the **conjugate** $a - \sqrt{b}$).

> EXAMPLE
> $$\frac{1}{3 + \sqrt{2}} = \frac{3 - \sqrt{2}}{9 - 2} = \frac{3 - \sqrt{2}}{7}.$$

Reciprocal

The reciprocal of $x \neq 0$ is $\frac{1}{x}$; a number times its reciprocal is 1.

Standard form

A number written as $A \times 10^n$ with $1 \leq A < 10$ and n an integer.

> EXAMPLE
> $0.00042 = 4.2 \times 10^{-4}$.

INEQUALITIES

More in **Algebraic methods**

Less than or equal to, greater than or equal to

$\leq$, $\geq$

SAY *is at most; is at least*

$x \leq 3$ allows $x = 3$; $x < 3$ does not. British exam papers often use the slanted forms $\leqslant$ and $\geqslant$, which mean the same.

> EXAMPLE
> $x^2 \leq 9 \iff -3 \leq x \leq 3$.

Much less than, much greater than

$\ll$, $\gg$

SAY *is much less than; is much greater than*

Used in modelling and approximation: one quantity is so much smaller that it can be ignored.

> EXAMPLE
> If $h \ll 1$ then h^2 is tiny compared with h, so $(1+h)^2 \approx 1 + 2h$.

More in **Binomial expansion**

Strict inequality

One using $<$ or $>$, so equality is not allowed. A non-strict inequality uses $\leq$ or $\geq$.

Rules for inequalities

You can add or subtract anything on both sides. Multiplying or dividing by a **negative** number reverses the inequality sign.

> EXAMPLE
> $-2x < 6 \iff x > -3$.

Critical values

The values where the two sides of an inequality are equal. For a quadratic inequality, find them first and then decide which side of each the solution lies on, usually from a sketch.

> EXAMPLE
> $x^2 - x - 6 > 0$: critical values -2 and 3, solution $x < -2$ or $x > 3$.

Region

The set of points in the plane satisfying one or more inequalities, shaded on a graph. A dashed boundary means it is not included (strict inequality); a solid one means it is.

POLYNOMIALS AND ALGEBRAIC FRACTIONS

More in **Polynomials**

Polynomial

An expression $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ in which every power of x is a non-negative integer.

> EXAMPLE
> $4x^3 - x + 2$ is a polynomial; $\sqrt{x}$ and $\frac{1}{x}$ are not.

Degree

The highest power of the variable in a polynomial with a non-zero coefficient. Linear, quadratic, cubic and quartic mean degree 1, 2, 3 and 4.

Leading coefficient

The coefficient of the highest power. A polynomial whose leading coefficient is 1 is called **monic**.

Factor theorem

$(x-a)$ is a factor of the polynomial $f(x)$ if and only if $f(a)=0$. More generally $(ax-b)$ is a factor if and only if $f\left(\frac{b}{a}\right)=0$.

> EXAMPLE
> $f(x)=x^3-7x+6$ has $f(1)=0$, so $(x-1)$ is a factor.

Remainder theorem

When the polynomial $f(x)$ is divided by $(x-a)$, the remainder is $f(a)$.

Algebraic division

Dividing one polynomial by another to give a quotient and a remainder: $f(x)=(\text{divisor})(\text{quotient})+\text{remainder}$, where the remainder has lower degree than the divisor.

Algebraic fraction

A fraction whose numerator and denominator are algebraic expressions. Simplify by factorising and cancelling common **factors** (never terms).

Proper and improper algebraic fractions

A fraction of polynomials is proper if the degree of the numerator is less than that of the denominator, and improper otherwise. An improper one can be divided out first.

> EXAMPLE
> $\frac{x^2+1}{x-1}$ is improper; it equals $x+1+\frac{2}{x-1}$.

More in **Partial fractions**

Partial fractions

Splitting a fraction with a factorised denominator into a sum of simpler fractions, one for each factor. Used for integration and for series expansions.

> EXAMPLE
> $$\frac{5}{(x-2)(x+3)}=\frac{1}{x-2}-\frac{1}{x+3}.$$

More in **Partial fractions**

Roots and coefficients

For a polynomial with roots $\alpha,\beta,\ldots$, the coefficients give the sums and products of the roots. For ax^2+bx+c: $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$.

> EXAMPLE
> For ax^3+bx^2+cx+d: $\sum\alpha=-\frac{b}{a}$, $\sum\alpha\beta=\frac{c}{a}$, $\alpha\beta\gamma=-\frac{d}{a}$.

More in **Roots of polynomials and series**

COUNTING AND THE BINOMIAL EXPANSION

More in **Binomial expansion**

Factorial

$n!$

SAY *n factorial*

$n! = n \times (n-1) \times \cdots \times 2 \times 1$, the number of ways to arrange n different objects in a row. By definition $0! = 1$ (there is one way to arrange nothing).

> EXAMPLE
> $5! = 120$.

Binomial coefficient

$\binom{n}{r}$

SAY *n choose r*

The number of ways to choose r objects from n when order does not matter: $\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$. Also written ${}^{n}C_{r}$ or ${}_{n}C_{r}$.

> EXAMPLE
> $\binom{5}{2} = 10$, so $(1+x)^5$ has $10x^2$ as its x^2 term.

Permutations

${}^{n}P_{r}$

SAY *n P r*

The number of ways to choose r objects from n **in order**: $\frac{n!}{(n-r)!}$.

> EXAMPLE
> ${}^{5}P_{2} = 20$: first and second place from 5 runners.

More in **Probability**

Permutation

An arrangement in which order matters. There are $n!$ ways to arrange n different objects in a row.

> EXAMPLE
> The letters of CAT can be arranged in $3! = 6$ ways.

Combination

A selection in which order does not matter. There are $\binom{n}{r}$ ways to choose r objects from n.

> EXAMPLE
> A team of 3 from 10 people: $\binom{10}{3} = 120$.

Pascal's triangle

The triangle of numbers in which each entry is the sum of the two above it. Row n (starting from row 0) lists $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$.

> EXAMPLE
> Row 4 is $1, 4, 6, 4, 1$.

Binomial expansion

For a positive integer n: $(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$. For other n, $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots$ is an infinite series valid when $|x| < 1$.

> EXAMPLE
> $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$ for $|x| < 1$.

Range of validity

The values of x for which an infinite series expansion converges to the function. For $(a+bx)^n$ with n not a positive integer, it is $\left|\frac{bx}{a}\right| < 1$.

Functions and graphs

What a function is, the words for describing one, and how graphs are transformed.

FUNCTIONS

More in **Functions and graphs**

A function from A to B

$f : A \to B$

SAY *f from A to B*

f assigns to every element of the domain A exactly one element of the codomain B. The range (or image) is the set of outputs actually reached, which can be smaller than B.

> EXAMPLE
> $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ has codomain $\mathbb{R}$ but range $[0, \infty)$.

Maps to

$x \mapsto f(x)$

SAY *x maps to …*

Describes what a function does to a single input, without naming the function.

> EXAMPLE
> $x \mapsto 2x + 1$ doubles and adds one, so $3 \mapsto 7$.

Mapping

A rule that sends each input to one or more outputs. A function is a mapping that sends each input to **exactly one** output.

Domain

The set of inputs a function is defined on. It is part of the function: changing the domain gives a different function.

> EXAMPLE
> $f(x) = \sqrt{x - 3}$ needs the domain $x \geq 3$.

Range

The set of outputs a function actually takes, as x runs over the domain. It is also called the image.

> EXAMPLE
> $f(x) = x^2 + 1$, $x \in \mathbb{R}$, has range $f(x) \geq 1$.

Codomain

The set the outputs are declared to lie in, written after the arrow in $f : A \to B$. The range is the part of the codomain actually reached.

One-to-one function

A function in which different inputs always give different outputs. Only one-to-one functions have inverses. (At university: injective.)

> EXAMPLE
> $f(x) = x^3$ is one-to-one; $f(x) = x^2$ on $\mathbb{R}$ is not, since $f(-2) = f(2)$.

Many-to-one function

A function in which some output comes from more than one input.

Composite function

$fg(x)$, $(f \circ g)(x)$

SAY *f of g of x; f after g*

$f(g(x))$: apply g first, then f. A level writes $fg(x)$; university writes $(f \circ g)(x)$. The order matters.

> EXAMPLE
> With $f(x) = x^2$ and $g(x) = x + 1$: $fg(x) = (x+1)^2$ but $gf(x) = x^2 + 1$.

Inverse function

f^{-1}

SAY *f inverse*

The function that undoes f: $f^{-1}(f(x)) = x$. It is **not** $\frac{1}{f(x)}$. Only one-to-one functions have inverses; its graph is the reflection of $y = f(x)$ in $y = x$.

> EXAMPLE
> If $f(x) = 2x + 1$ then $f^{-1}(x) = \frac{x-1}{2}$.

Self-inverse function

A function that is its own inverse: $f(f(x)) = x$. Its graph is symmetric in $y = x$.

> EXAMPLE
> $f(x) = \frac{1}{x}$ and $f(x) = 5 - x$.

Even function

A function with $f(-x) = f(x)$ for every x. Its graph is symmetric in the y-axis.

> EXAMPLE
> x^2, $\cos x$, $|x|$.

Odd function

A function with $f(-x) = -f(x)$ for every x. Its graph has rotational symmetry of order 2 about the origin.

> EXAMPLE
> x^3, $\sin x$, $\tan x$.

Periodic function

A function with $f(x + T) = f(x)$ for every x, for some $T > 0$. The smallest such T is the period.

> EXAMPLE
> $\sin x$ has period 2π; $\tan x$ has period π.

Increasing and decreasing functions

f is increasing on an interval if $x_1 < x_2$ implies $f(x_1) \leq f(x_2)$ (strictly increasing with $<$). For a differentiable function, $f'(x) > 0$ throughout means strictly increasing.

Piecewise function

A function defined by different rules on different parts of its domain.

> EXAMPLE
> $$f(x) = \begin{cases} x^2 & x < 0 \\ 2x & x \geq 0 \end{cases}$$

Modulus (absolute value)

$|x|$

SAY *mod x; the absolute value of x*

The size of x ignoring its sign: $|x| = x$ if $x \geq 0$ and $-x$ if $x < 0$. Think of $|a - b|$ as the distance between a and b.

> EXAMPLE
> $|-3| = 3$, and $|x - 2| < 1 \iff 1 < x < 3$.

Floor and ceiling

$\lfloor x \rfloor$, $\lceil x \rceil$

SAY *floor of x; ceiling of x*

$\lfloor x \rfloor$ is the largest integer $\leq x$ (round down); $\lceil x \rceil$ is the smallest integer $\geq x$ (round up). Watch negative numbers.

> EXAMPLE
> $\lfloor 2.7 \rfloor = 2$, $\lfloor -2.7 \rfloor = -3$ and $\lceil 2.1 \rceil = 3$.

GRAPHS AND TRANSFORMATIONS

More in **Functions and graphs**

Intercept

Where a graph crosses an axis. The y-intercept is at $x = 0$; the x-intercepts are the real roots.

Asymptote

A line that a curve approaches more and more closely, without necessarily ever reaching it. It can be vertical, horizontal or oblique (slanted).

> EXAMPLE
> $y = \frac{1}{x-2}$ has asymptotes $x = 2$ and $y = 0$.

Translation

A shift without turning or stretching. $y = f(x) + a$ moves the graph up by a; $y = f(x + a)$ moves it **left** by a. As a vector: $\begin{pmatrix} -a \\ 0 \end{pmatrix}$.

Stretch

$y = af(x)$ stretches the graph vertically by scale factor a. $y = f(ax)$ stretches it horizontally by scale factor $\frac{1}{a}$.

Reflection of a graph

$y = -f(x)$ reflects the graph in the x-axis; $y = f(-x)$ reflects it in the y-axis; the inverse function's graph is the reflection in $y = x$.

Modulus graphs

$y = |f(x)|$ reflects the parts of the graph below the x-axis upwards. $y = f(|x|)$ keeps the part for $x \geq 0$ and reflects it in the y-axis.

Reciprocal graph

The graph of $y = \frac{1}{f(x)}$. Roots of f become vertical asymptotes, and large values become small ones.

Sketch

A graph drawn by hand showing the correct shape, the intercepts, turning points and asymptotes, with their coordinates or equations labelled. It does not need to be to scale.

Gradient of a line

$m = \dfrac{y_2 - y_1}{x_2 - x_1}$, the change in y for each unit increase in x.

More in **Straight lines and circles**

EXPONENTIALS AND LOGARITHMS

More in **Exponentials and logarithms**

Exponential function

A function $f(x) = a^x$ with base $a > 0$, $a \neq 1$. Its graph passes through $(0, 1)$ and has the x-axis as an asymptote.

Euler's number

e

SAY *e*

The constant $\mathrm{e} \approx 2.71828$, chosen so that $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x = \mathrm{e}^x$. $\exp(x)$ is another way to write e^x, handy when the power is long.

> EXAMPLE
> $\mathrm{e} = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$.

Exponential growth and decay

A quantity grows exponentially when its rate of increase is proportional to its size: $N = N_0 e^{kt}$ with $k > 0$. With $k < 0$ it decays.

Half-life

The time taken for an exponentially decaying quantity to halve. For $N = N_0 e^{-kt}$ it is $\frac{\ln 2}{k}$.

Logarithm

$\log_a x$ is the power you raise a to in order to get x: $\log_a x = y \iff a^y = x$.

> EXAMPLE
> $\log_2 32 = 5$ since $2^5 = 32$.

Logarithm to base a

$\log_a x$

SAY *log to base a of x*

The power you raise a to in order to get x. Plain $\log x$ is ambiguous: base 10 on calculators and in school, but usually base e at university.

> EXAMPLE
> $\log_2 8 = 3$ because $2^3 = 8$.

Natural logarithm

$\ln x$

SAY *"log x" or "lin x"*

The logarithm to base e: $\ln x = y \iff \mathrm{e}^y = x$. Defined for $x > 0$ only.

> EXAMPLE
> $\ln 1 = 0$, $\ln \mathrm{e} = 1$ and $\mathrm{e}^{\ln x} = x$.

Laws of logarithms

$\log_a(xy) = \log_a x + \log_a y$, $\log_a \frac{x}{y} = \log_a x - \log_a y$, $\log_a x^k = k \log_a x$, and the change of base rule $\log_a x = \frac{\log_b x}{\log_b a}$.

Reducing to linear form

Taking logs turns $y = ax^n$ into $\log y = n \log x + \log a$, a straight line in $(\log x, \log y)$, and $y = kb^x$ into $\log y = x \log b + \log k$, a straight line in $(x, \log y)$.

Trigonometry

Angles, the trigonometric and hyperbolic functions, and the identities that connect them.

ANGLES AND MEASURES

More in **Trigonometry**

Radian

The angle at the centre of a circle subtended by an arc equal in length to the radius. π radians $= 180^\circ$, so 1 radian is about 57.3°.

Arc length

The length of part of a circle's circumference: $s = r\theta$ with θ in radians.

Sector and segment

A sector is the region between two radii and an arc, area $\frac{1}{2}r^2\theta$. A segment is the region between a chord and an arc, area $\frac{1}{2}r^2(\theta - \sin\theta)$.

Unit circle

The circle of radius 1 centred at the origin. The point at angle θ from the positive x-axis is $(\cos\theta, \sin\theta)$, which defines $\sin$ and $\cos$ for every angle.

CAST diagram

A memory aid for the signs of the trig functions in the four quadrants: All positive in the first, Sine in the second, Tangent in the third, Cosine in the fourth.

Exact values

The values of sin, cos and tan at 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$ and $\frac{\pi}{2}$ that can be written with surds.

> EXAMPLE
> $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, $\tan\frac{\pi}{4} = 1$.

Small-angle approximations

For small θ in radians: $\sin\theta \approx \theta$, $\tan\theta \approx \theta$, $\cos\theta \approx 1 - \frac{\theta^2}{2}$.

Bearing

A direction given as an angle measured clockwise from north, written with three figures.

> EXAMPLE
> South-west is 225°.

Angle of elevation and depression

The angle above the horizontal (elevation) or below it (depression) of the line of sight to an object.

TRIGONOMETRIC FUNCTIONS

More in **Trigonometry**

Sine, cosine and tangent

sin, cos, tan

In a right-angled triangle: $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$, $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$, $\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sin\theta}{\cos\theta}$.

Sine rule

In any triangle, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$. With two angles and a side, or two sides and a non-included angle (which can give two triangles: the **ambiguous case**).

Cosine rule

In any triangle, $a^2 = b^2 + c^2 - 2bc\cos A$. Use it with three sides, or two sides and the angle between them.

Area of a triangle

$\text{Area} = \frac{1}{2}ab\sin C$, where C is the angle between sides a and b.

Powers of trig functions (a trap)

$\sin^2 x$, $\sin^{-1} x$

SAY *sine squared x; inverse sine of x*

A clash of conventions: $\sin^2 x$ means $(\sin x)^2$, but $\sin^{-1} x$ means the **inverse** function $\arcsin x$, not $\frac{1}{\sin x}$ (that is $\operatorname{cosec} x$).

> EXAMPLE
> $\sin^2 \frac{\pi}{6} = \frac{1}{4}$, while $\sin^{-1} \frac{1}{2} = \frac{\pi}{6}$.

Reciprocal trig functions

$\sec$, cosec, $\cot$

SAY *secant; cosecant; cotangent*

$\sec x = \frac{1}{\cos x}$, $\operatorname{cosec} x = \frac{1}{\sin x}$ (written $\csc x$ in America) and $\cot x = \frac{1}{\tan x}$. A useful memory aid: the third letter of each name tells you which function it is the reciprocal of.

> EXAMPLE
> $1 + \tan^2 x \equiv \sec^2 x$.

More in **Trigonometric identities and equations**

Inverse sine

$\arcsin x$

SAY *arc sine of x*

The angle in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ whose sine is x. Similarly $\arccos x \in [0, \pi]$ and $\arctan x \in (-\frac{\pi}{2}, \frac{\pi}{2})$. These restricted ranges make them functions.

> EXAMPLE
> $\arcsin \frac{1}{2} = \frac{\pi}{6}$, even though $\sin \frac{5\pi}{6}$ is also $\frac{1}{2}$.

Principal value

The single value an inverse trig function returns. $\arcsin$ and $\arctan$ give values in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\arccos$ in $[0, \pi]$; other solutions of an equation are found from these using symmetry and the period.

Amplitude

The largest distance of a wave from its mean level. $y = a\sin x$ has amplitude $|a|$.

TRIGONOMETRIC IDENTITIES

More in **Trigonometric identities and equations**

Pythagorean identities

$\sin^2\theta + \cos^2\theta \equiv 1$, and dividing through, $1 + \tan^2\theta \equiv \sec^2\theta$ and $1 + \cot^2\theta \equiv \operatorname{cosec}^2\theta$.

Compound angle formulae

$\sin(A \pm B) \equiv \sin A \cos B \pm \cos A \sin B$, $\cos(A \pm B) \equiv \cos A \cos B \mp \sin A \sin B$ and $\tan(A \pm B) \equiv \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$.

Double angle formulae

$\sin 2A \equiv 2 \sin A \cos A$, $\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$ and $\tan 2A \equiv \dfrac{2\tan A}{1 - \tan^2 A}$.

Harmonic form (R cos form)

Writing $a\cos\theta + b\sin\theta$ as $R\cos(\theta - \alpha)$ (or a similar form), with $R = \sqrt{a^2 + b^2}$ and $\tan\alpha = \frac{b}{a}$. It makes the maximum, minimum and solutions easy to read off.

> EXAMPLE
> $3\cos\theta + 4\sin\theta = 5\cos(\theta - 0.927\ldots)$, so its maximum is 5.

The t-formulae

With $t = \tan\frac{\theta}{2}$: $\sin\theta = \dfrac{2t}{1+t^2}$, $\cos\theta = \dfrac{1-t^2}{1+t^2}$, $\tan\theta = \dfrac{2t}{1-t^2}$.

Factor formulae

Sums of sines and cosines as products, such as $\sin P + \sin Q \equiv 2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2}$.

HYPERBOLIC FUNCTIONS

More in **Hyperbolic functions**

Hyperbolic functions

sinh, cosh, tanh

SAY *"shine" or "sinch"; "cosh"; "than" or "tanch"*

$\sinh x = \frac{e^x - e^{-x}}{2}$, $\cosh x = \frac{e^x + e^{-x}}{2}$, $\tanh x = \frac{\sinh x}{\cosh x}$. They play the role for the hyperbola $x^2 - y^2 = 1$ that $\sin$ and $\cos$ play for the circle.

> EXAMPLE
> $\cosh^2 x - \sinh^2 x \equiv 1$.

Inverse hyperbolic functions

arsinh x

SAY *"ar-shine" x*

Inverses of the hyperbolic functions, also written $\sinh^{-1} x$. They can be written with logarithms. The prefix "ar" stands for *area*, not *arc*.

> EXAMPLE
> $\operatorname{arsinh} x = \ln\left(x + \sqrt{x^2 + 1}\right)$.

Hyperbolic identities

$\cosh^2 x - \sinh^2 x \equiv 1$, $\sinh 2x \equiv 2\sinh x \cosh x$ and $\cosh 2x \equiv \cosh^2 x + \sinh^2 x$.

Osborn's rule

A trig identity becomes the matching hyperbolic identity if you replace each trig function by its hyperbolic version and change the sign of every product (or implied product) of two sines.

> EXAMPLE
> $\cos^2 x + \sin^2 x \equiv 1$ becomes $\cosh^2 x - \sinh^2 x \equiv 1$.

Inverse hyperbolic functions as logarithms

$\operatorname{arsinh} x = \ln\left(x + \sqrt{x^2+1}\right)$, $\operatorname{arcosh} x = \ln\left(x + \sqrt{x^2-1}\right)$ for $x \geq 1$, $\operatorname{artanh} x = \frac{1}{2}\ln\frac{1+x}{1-x}$ for $|x| < 1$.

Coordinate geometry

Lines, circles and curves described by equations, in Cartesian, parametric and polar form.

STRAIGHT LINES

More in **Straight lines and circles**

Cartesian coordinates

A point's position given as (x, y): its distances along the x-axis and y-axis from the origin.

Equation of a straight line

$y = mx + c$ (gradient m, y-intercept c), $y - y_1 = m(x - x_1)$ (through (x_1, y_1)), or $ax + by + c = 0$.

Midpoint

The point halfway between (x_1, y_1) and (x_2, y_2): $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

Distance between two points

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, by Pythagoras.

Parallel lines

Lines with equal gradients.

Perpendicular lines

Lines at right angles. Their gradients multiply to -1: $m_1 m_2 = -1$.

> EXAMPLE
> Perpendicular to gradient $\frac{2}{3}$ is gradient $-\frac{3}{2}$.

Perpendicular bisector

The line through the midpoint of a segment at right angles to it. Every point on it is the same distance from both ends.

Collinear

Three or more points are collinear if they lie on one straight line.

CIRCLES

More in **Straight lines and circles**

Equation of a circle

$(x - a)^2 + (y - b)^2 = r^2$ is the circle with centre (a, b) and radius r. The expanded form $x^2 + y^2 + 2fx + 2gy + c = 0$ has centre $(-f, -g)$.

Chord

A straight line segment joining two points on a curve. The perpendicular bisector of a chord of a circle passes through the centre.

Tangent to a circle

A line touching the circle at exactly one point. It is perpendicular to the radius at that point.

Diameter

A chord through the centre. The angle in a semicircle is a right angle (Thales).

Circumcircle

The circle through all three vertices of a triangle. Its centre is where the perpendicular bisectors of the sides meet.

PARAMETRIC EQUATIONS

More in **Parametric equations**

Parameter

A third variable (often t or θ) used to describe both coordinates of a curve: $x = f(t)$, $y = g(t)$.

Parametric equations

A pair $x = f(t)$, $y = g(t)$ giving a point of the curve for each value of the parameter.

> EXAMPLE
> $x = 3\cos t$, $y = 3\sin t$ is the circle of radius 3.

Cartesian equation

An equation connecting x and y directly, with no parameter. Found by eliminating the parameter.

> EXAMPLE
> $x = t^2$, $y = 2t$ gives $y^2 = 4x$.

Parametric differentiation

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t}$, valid where $\frac{\mathrm{d}x}{\mathrm{d}t} \neq 0$.

POLAR COORDINATES

More in **Polar coordinates**

Polar coordinates

A point's position given as (r, θ): its distance r from the pole and the angle θ anticlockwise from the initial line. $x = r\cos\theta$, $y = r\sin\theta$ and $r^2 = x^2 + y^2$.

Pole and initial line

The pole is the origin of polar coordinates; the initial line is the half-line from it where $\theta = 0$, usually the positive x-axis.

Area in polar coordinates

The area swept out by a polar curve from $\theta = \alpha$ to $\theta = \beta$ is $\frac{1}{2}\int_{\alpha}^{\beta} r^2 \,\mathrm{d}\theta$.

Cardioid

The heart-shaped curve $r = a(1 + \cos\theta)$.

CONIC SECTIONS

More in **Roots of polynomials and series**

Conic section

A curve formed where a plane cuts a double cone: a circle, ellipse, parabola or hyperbola.

Parabola

The set of points equidistant from a fixed point (the **focus**) and a fixed line (the **directrix**). Standard form $y^2 = 4ax$, with focus $(a, 0)$ and directrix $x = -a$.

Ellipse

A stretched circle, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. The sum of the distances from any point on it to the two foci is constant.

Hyperbola

The curve $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, with two branches and asymptotes $y = \pm \frac{b}{a} x$. The rectangular hyperbola $xy = c^2$ has perpendicular asymptotes.

Eccentricity

A number $e \geq 0$ measuring how far a conic is from a circle: $e = 0$ circle, $0 < e < 1$ ellipse, $e = 1$ parabola, $e > 1$ hyperbola.

Sequences and series

Lists of numbers, their sums, what happens in the long run, and proof by induction.

SEQUENCES

More in **Sequences and series**

Sequence

An ordered list of numbers $u_1, u_2, u_3, \ldots$ (its terms). It can be finite or infinite.

The nth term of a sequence

u_n

SAY *u n; u sub n*

The term in position n. The whole sequence is written (u_n) or $\{u_n\}$. A recurrence relation such as $u_{n+1} = 2u_n + 1$ defines each term from the previous one.

> EXAMPLE
> If $u_n = 3n - 1$, then $u_1 = 2$, $u_2 = 5$, $u_{10} = 29$.

And so on (ellipsis)

$\ldots, \cdots$

SAY *dot dot dot; and so on*

The pattern continues. Low dots $\ldots$ go between commas; centred dots $\cdots$ go between operations such as $+$ and $\times$.

> EXAMPLE
> $1, 2, \ldots, n$ and $1 + 2 + \cdots + n$.

Recurrence relation

A rule giving each term from the one (or ones) before, together with a starting value.

> EXAMPLE
> $u_{n+1} = 2u_n + 1$, $u_1 = 1$ gives $1, 3, 7, 15, \ldots$

Arithmetic sequence

A sequence with a constant difference d between consecutive terms: $u_n = a + (n-1)d$.

> EXAMPLE
> $5, 8, 11, 14, \ldots$ has $a = 5$, $d = 3$.

Geometric sequence

A sequence with a constant ratio r between consecutive terms: $u_n = ar^{n-1}$.

> EXAMPLE
> $3, 6, 12, 24, \ldots$ has $a = 3$, $r = 2$.

Increasing and decreasing sequences

A sequence is increasing if $u_{n+1} > u_n$ for every n and decreasing if $u_{n+1} < u_n$ for every n. Either is called **monotonic**.

Periodic sequence

A sequence whose terms repeat in a cycle. The length of the cycle is the order (or period).

> EXAMPLE
> $u_{n+1} = \frac{1}{1-u_n}$, $u_1 = 2$ gives $2, -1, \frac{1}{2}, 2, \ldots$, period 3.

Convergent and divergent sequences

A sequence converges if its terms get as close as you like to a fixed number L (its limit) and stay there. Otherwise it diverges, for instance by growing without bound or oscillating.

> EXAMPLE
> $\frac{1}{n} \to 0$ converges; $(-1)^n$ and n^2 diverge.

Tends to (or maps to a set)

$\to$

SAY *tends to; approaches*

In limits, $x \to a$ means x gets arbitrarily close to a. In function notation, $f : A \to B$ means f takes inputs from A and gives outputs in B.

> EXAMPLE
> As $x \to \infty$, $\frac{1}{x} \to 0$.

More in **Differentiation**

Limit

$\lim_{n\to\infty}$

SAY *the limit as n tends to infinity of …*

The value a sequence or function gets arbitrarily close to. It can exist even when the value at that point does not, as in $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

> EXAMPLE
> $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \mathrm{e}$.

Fibonacci sequence

$1, 1, 2, 3, 5, 8, \ldots$, defined by $F_{n+2} = F_{n+1} + F_n$. The ratio of consecutive terms tends to the golden ratio $\frac{1+\sqrt{5}}{2}$.

SERIES

More in **Sequences and series**

Series

The sum of the terms of a sequence.

Sum (sigma notation)

$\sum_{r=1}^{n} a_r$

SAY *the sum from r equals 1 to n of a r*

Add up a_r for every whole number r from the bottom limit to the top limit. The letter r is a dummy variable: using k instead changes nothing.

> EXAMPLE
> $\sum_{r=1}^{4} r^2 = 1 + 4 + 9 + 16 = 30$.

Product (pi notation)

$\prod_{r=1}^{n} a_r$

SAY *the product from r equals 1 to n of a r*

Multiply a_r together for every r from the bottom limit to the top. An empty product is 1, just as an empty sum is 0.

> EXAMPLE
> $\prod_{r=1}^{4} r = 24 = 4!$.

More in **Roots of polynomials and series**

Partial sum and sum to infinity

S_n, S_∞

SAY *S n; S infinity*

S_n is the sum of the first n terms. S_∞ is the limit of S_n as $n \to \infty$, when that limit exists.

> EXAMPLE
> For $1 + \frac{1}{2} + \frac{1}{4} + \cdots$: $S_n = 2 - \frac{1}{2^{n-1}}$ and $S_\infty = 2$.

Arithmetic series

The sum of an arithmetic sequence: $S_n = \frac{n}{2}\big(2a + (n-1)d\big) = \frac{n}{2}(a + l)$ where l is the last term.

> EXAMPLE
> $1 + 2 + \cdots + 100 = 50 \times 101 = 5050$.

Geometric series

The sum of a geometric sequence: $S_n = \dfrac{a(1 - r^n)}{1 - r}$ for $r \neq 1$.

Sum to infinity

S_∞

SAY *S infinity*

The limit of the partial sums S_n as $n \to \infty$, if it exists. For a geometric series it exists exactly when $|r| < 1$, and then $S_\infty = \dfrac{a}{1 - r}$.

> EXAMPLE
> $1 + \frac{1}{2} + \frac{1}{4} + \cdots = 2$.

Convergent series

A series whose partial sums tend to a finite limit. Terms tending to 0 is necessary but not sufficient.

> EXAMPLE
> $\sum \frac{1}{n}$ diverges (the harmonic series) although $\frac{1}{n} \to 0$.

Standard sums

$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$, $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$, $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$.

More in **Roots of polynomials and series**

Method of differences

Writing each term of a series as $f(r) - f(r+1)$, so that almost everything cancels in the sum (it **telescopes**).

> EXAMPLE
> $\sum_{r=1}^{n} \frac{1}{r(r+1)} = \sum \left(\frac{1}{r} - \frac{1}{r+1}\right) = 1 - \frac{1}{n+1}$.

More in **Roots of polynomials and series**

PROOF BY INDUCTION

More in **Mathematical induction**

Base case

The first step of an induction proof: checking the statement for the smallest value (often $n = 1$).

Inductive hypothesis

The assumption, in an induction proof, that the statement is true for $n = k$.

Inductive step

Proving that if the statement is true for $n = k$, then it is true for $n = k + 1$.

Conclusion of an induction proof

The closing sentence: since the statement is true for $n = 1$, and true for $n = k + 1$ whenever it is true for $n = k$, it is true for all $n \geq 1$. Exams give a mark for it.

Calculus

Differentiation, integration, differential equations and series expansions.

DIFFERENTIATION

More in **Differentiation**

Limit

$\lim_{x \to a} f(x) = L$ means $f(x)$ can be made as close as you like to L by taking x close enough to a (but not equal to it).

Derivative

The gradient of a curve at a point, and the instantaneous rate of change of y with respect to x: $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$.

Differentiation from first principles

Finding a derivative directly from the limit definition, by expanding $f(x+h) - f(x)$, dividing by h and letting $h \to 0$.

Derivative (Leibniz notation)

$\frac{\mathrm{d}y}{\mathrm{d}x}$

SAY *dee y by dee x*

The rate of change of y with respect to x: the gradient of the curve. It is not really a fraction, but it often behaves like one, as in the chain rule $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$.

> EXAMPLE
> If $y = x^3$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2$.

Derivative (Lagrange notation)

$f'(x)$, $f''(x)$

SAY *f dash x; f double dash x (America: "f prime")*

The derivative of the function f, then the second derivative. For higher derivatives write $f^{(n)}(x)$, with the brackets showing it is not a power.

> EXAMPLE
> If $f(x) = \sin x$ then $f'(x) = \cos x$ and $f''(x) = -\sin x$.

Second derivative

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$

SAY *dee two y by dee x squared*

The derivative of the derivative: how fast the gradient is changing. Its sign tells you whether a curve is concave up or down. Note where the 2s go.

> EXAMPLE
> If $y = x^3$ then $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6x$.

The nth derivative

$f^{(n)}(x)$

SAY *the nth derivative of f*

Differentiate f a total of n times. Used in Maclaurin and Taylor series.

> EXAMPLE
> If $f(x) = \mathrm{e}^{2x}$ then $f^{(n)}(x) = 2^n \mathrm{e}^{2x}$.

More in **Further calculus**

Time derivatives (Newton notation)

$\dot{x}$, $\ddot{x}$

SAY *x dot; x double dot*

A dot means differentiate with respect to **time**. Common in mechanics and differential equations.

> EXAMPLE
> If x is position, $\dot{x}$ is velocity and $\ddot{x}$ is acceleration.

More in **Kinematics**

A small change in x

δx

SAY *delta x*

A small (but non-zero) increase in x. Derivatives arise as $\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$. Some books write h instead.

Change in

Δ

SAY *delta*

The total change in a quantity: final value minus initial value. Unlike δ, it need not be small.

> EXAMPLE
> If speed rises from u to v, then $\Delta v = v - u$.

More in **Kinematics**

Differentiable

A function is differentiable at a point if its derivative exists there: the graph has a single, non-vertical tangent. A corner (like $|x|$ at 0) is not differentiable.

Tangent

The straight line that touches a curve at a point and has the same gradient as the curve there.

Normal

The line through a point of a curve perpendicular to the tangent there. Its gradient is $-\frac{1}{f'(x)}$.

Chain rule

For a composite function: $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}y}{\mathrm{d}u} \times \dfrac{\mathrm{d}u}{\mathrm{d}x}$.

> EXAMPLE
> $\frac{\mathrm{d}}{\mathrm{d}x}(3x+1)^5 = 15(3x+1)^4$.

Product rule

$$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}.$$

Quotient rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right) = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}.$$

Implicit differentiation

Differentiating an equation in x and y term by term without first making y the subject, using the chain rule: $\frac{\mathrm{d}}{\mathrm{d}x}\big(g(y)\big) = g'(y)\frac{\mathrm{d}y}{\mathrm{d}x}$.

> EXAMPLE
> $x^2 + y^2 = 25$ gives $2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.

Connected rates of change

Linking the rates of change of related quantities with the chain rule.

> EXAMPLE
> For a sphere, $\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}t} = 4\pi r^2\frac{\mathrm{d}r}{\mathrm{d}t}$.

STATIONARY POINTS AND THE SHAPE OF A CURVE

More in **Differentiation**

Stationary point

A point where $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$: the tangent is horizontal. It is a local maximum, a local minimum or a stationary point of inflection.

Turning point

A stationary point where the gradient changes sign: a local maximum or a local minimum.

Local maximum and minimum

A local maximum is higher than all nearby points of the curve; a local minimum is lower. The **global** maximum or minimum is the highest or lowest value over the whole domain.

Second derivative test

At a stationary point, $f''(x) > 0$ means a minimum and $f''(x) < 0$ a maximum. If $f''(x) = 0$ the test fails: check the gradient on either side.

Concave and convex

A curve is convex (concave up) where $f''(x) > 0$, bending like a cup, and concave (concave down) where $f''(x) < 0$.

Point of inflection

A point where the curve changes between concave and convex, so $f''(x)$ changes sign. It need not be stationary.

> EXAMPLE
> $y = x^3$ has a stationary point of inflection at the origin.

Optimisation

Finding the largest or smallest value of a quantity, usually by writing it as a function of one variable and finding its stationary points.

INTEGRATION

More in **Integration techniques**

Integral (antiderivative)

A function whose derivative is the given function. Integration is the reverse of differentiation.

Indefinite integral

$\int f(x)\,\mathrm{d}x$

SAY *the integral of f of x with respect to x*

The family of all functions whose derivative is $f(x)$, so you add a constant $+c$. The $\mathrm{d}x$ says which variable you are integrating with respect to; it is not decoration.

> EXAMPLE
> $\int 2x\,\mathrm{d}x = x^2 + c$.

Constant of integration

$+c$

The $+c$ in an indefinite integral: every antiderivative of a function differs from any other by a constant.

Definite integral

$\int_a^b f(x)\,\mathrm{d}x$

SAY *the integral from a to b of f of x dee x*

A number: the signed area between the curve and the x-axis from $x = a$ to $x = b$ (area below the axis counts as negative).

> EXAMPLE
> $\int_0^1 x^2\,\mathrm{d}x = \frac{1}{3}$.

Evaluate between limits

$\left[F(x)\right]_a^b$

SAY *F of x between a and b*

Shorthand for $F(b) - F(a)$, used in the middle of working out a definite integral.

> EXAMPLE
> $\left[x^2\right]_1^3 = 9 - 1 = 8$.

Fundamental theorem of calculus

If $F' = f$ then $\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$: the area under a curve can be found from any antiderivative.

Area under a curve

$\int_a^b f(x)\,\mathrm{d}x$ gives the area between $y = f(x)$ and the x-axis when $f(x) \geq 0$. Regions below the axis count as negative, so find them separately.

Area between curves

$\int_a^b \big(f(x) - g(x)\big)\,\mathrm{d}x$ where $f(x) \geq g(x)$ on $[a, b]$ (top curve minus bottom curve).

Reverse chain rule (integration by inspection)

Recognising an integrand as the result of the chain rule, such as $\int f'(x)\big(f(x)\big)^n\,\mathrm{d}x = \frac{(f(x))^{n+1}}{n+1} + c$ and $\int \frac{f'(x)}{f(x)}\,\mathrm{d}x = \ln|f(x)| + c$.

Integration by substitution

Changing the variable, $u = g(x)$, to turn the integral into a simpler one. Remember to replace $\mathrm{d}x$ and, for a definite integral, the limits.

Integration by parts

$\int u\dfrac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\dfrac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$, the reverse of the product rule.

> EXAMPLE
> $\int xe^x\,\mathrm{d}x = xe^x - e^x + c$.

Volume of revolution

Rotating $y = f(x)$ from $x = a$ to $x = b$ through 2π about the x-axis gives volume $\pi\int_a^b y^2\,\mathrm{d}x$.

More in **Further calculus**

Mean value of a function

The average height of f over $[a, b]$: $\dfrac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$.

More in **Further calculus**

Improper integral

An integral with an infinite limit, or an integrand that is infinite somewhere in the range. It is defined as a limit, and converges if that limit exists.

> EXAMPLE
> $\int_1^\infty \frac{1}{x^2}\,\mathrm{d}x = \lim_{b\to\infty}\left(1 - \frac{1}{b}\right) = 1$.

More in **Further calculus**

DIFFERENTIAL EQUATIONS

More in **Differential equations**

Differential equation

An equation involving derivatives of an unknown function. Its **order** is that of the highest derivative.

General solution

The solution with all its arbitrary constants (one for each order), giving the whole family of solution curves.

Particular solution

The one solution that also satisfies given conditions, such as an initial value $y(0) = 2$, which fix the constants.

Separable differential equation

One that can be written $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y)$, and solved by $\int \frac{1}{g(y)}\,\mathrm{d}y = \int f(x)\,\mathrm{d}x$.

More in **Integration techniques**

Linear differential equation

One in which y and its derivatives appear only to the first power and are not multiplied together, such as $\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$.

Integrating factor

For $\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$, multiplying through by $e^{\int P\,\mathrm{d}x}$ turns the left side into $\frac{\mathrm{d}}{\mathrm{d}x}\left(ye^{\int P\,\mathrm{d}x}\right)$.

Homogeneous linear equation

A linear differential equation with zero on the right-hand side, such as $ay'' + by' + cy = 0$.

Auxiliary equation

For $ay'' + by' + cy = 0$, the quadratic $am^2 + bm + c = 0$. Its roots decide the form of the solution: $Ae^{m_1 x} + Be^{m_2 x}$, $(A + Bx)e^{mx}$, or $e^{px}(A\cos qx + B\sin qx)$.

Complementary function

The general solution of the homogeneous equation. The full solution is complementary function plus particular integral.

Particular integral

Any one solution of the full (non-homogeneous) equation, usually found by trying a function like the right-hand side.

Simple harmonic motion

Motion with acceleration proportional to displacement and directed towards a fixed point: $\ddot{x} = -\omega^2 x$, with solution $x = a\sin(\omega t + \varepsilon)$ and period $\frac{2\pi}{\omega}$.

Damping

A resistance that makes oscillations die away. Light damping still oscillates, critical damping returns fastest without oscillating, heavy damping returns slowly without oscillating.

Coupled differential equations

Two or more differential equations in which each unknown's rate depends on the others, such as predator–prey models.

SERIES EXPANSIONS

More in **Further calculus**

Maclaurin series

$f(x) = f(0) + f'(0)x + \dfrac{f''(0)}{2!}x^2 + \dfrac{f'''(0)}{3!}x^3 + \cdots$, a power series that matches f and all its derivatives at $x = 0$.

> EXAMPLE
> $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ for all x.

Taylor series

The same idea centred on $x = a$: $f(x) = f(a) + f'(a)(x-a) + \dfrac{f''(a)}{2!}(x-a)^2 + \cdots$

Power series

A series of the form $a_0 + a_1 x + a_2 x^2 + \cdots$, which may converge only for some values of x.

L'Hôpital's rule

If $f(a) = g(a) = 0$, then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$, when the second limit exists.

> EXAMPLE
> $\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\cos x}{1} = 1$.

Numerical methods

Finding roots and areas approximately, and measuring how accurate an answer is.

LOCATING AND FINDING ROOTS

More in **Numerical methods**

Change of sign

If f is continuous on $[a, b]$ and $f(a)$, $f(b)$ have opposite signs, there is a root between a and b. It can fail for a discontinuous function, and misses roots where the sign does not change.

Continuous function

A function whose graph has no breaks, jumps or holes over an interval: it can be drawn without lifting the pen.

Iteration

Repeating a process, feeding each output back in as the next input: $x_{n+1} = g(x_n)$.

Fixed-point iteration

Rearranging $f(x) = 0$ as $x = g(x)$ and iterating $x_{n+1} = g(x_n)$. It converges to a root α if $|g'(\alpha)| < 1$ (and you start close enough).

Staircase and cobweb diagrams

Pictures of fixed-point iteration on the graphs of $y = x$ and $y = g(x)$. A staircase approaches from one side ($0 < g' < 1$); a cobweb spirals in ($-1 < g' < 0$).

Newton–Raphson method

$x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$: follow the tangent at x_n down to the axis. Usually very fast, but it fails if $f'(x_n)$ is near 0.

Interval bisection

Repeatedly halving an interval that contains a sign change, keeping the half where the sign still changes.

More in **Further numerical methods**

Linear interpolation (false position)

Approximating a root by where the chord between $(a, f(a))$ and $(b, f(b))$ crosses the x-axis.

More in **Further numerical methods**

Order of convergence

How fast an iteration's error shrinks. Linear convergence multiplies the error by roughly a constant each step; Newton–Raphson is quadratic, roughly squaring it.

More in **Further numerical methods**

NUMERICAL INTEGRATION AND DIFFERENTIAL EQUATIONS

More in **Further numerical methods**

Trapezium rule

$\int_a^b y\,\mathrm{d}x \approx \frac{h}{2}\big(y_0 + 2(y_1 + \cdots + y_{n-1}) + y_n\big)$ with strips of width $h = \frac{b-a}{n}$. It overestimates for a convex curve and underestimates for a concave one.

More in **Numerical methods**

Ordinate

The y-value (height) of a curve at a given x. "$n+1$ ordinates" means n strips.

More in **Numerical methods**

Mid-ordinate rule

$\int_a^b y\,\mathrm{d}x \approx h\left(y_{\frac{1}{2}} + y_{\frac{3}{2}} + \cdots + y_{n-\frac{1}{2}}\right)$, using the heights at the middle of each strip.

Simpson's rule

$\int_a^b y\,\mathrm{d}x \approx \frac{h}{3}\big(y_0 + 4(y_1 + y_3 + \cdots) + 2(y_2 + y_4 + \cdots) + y_n\big)$ with an even number of strips. It fits parabolas instead of straight lines.

Euler's method

A step-by-step solution of $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$: $y_{n+1} = y_n + hf(x_n, y_n)$, following the tangent for a step of size h.

ACCURACY AND ERROR

More in **Numerical methods**

Decimal places and significant figures

Rounding to n decimal places keeps n digits after the point; to n significant figures keeps n digits counted from the first non-zero one.

> EXAMPLE
> 0.004718 is 0.0047 to 2 s.f. and 0.005 to 3 d.p.

Upper and lower bounds

The largest and smallest values a rounded number could have been. 7.4 to 1 d.p. lies in $[7.35, 7.45)$.

Absolute, relative and percentage error

Absolute error = approximation − exact value. Relative error = absolute error ÷ exact value; percentage error is that times 100.

Vectors

Quantities with size and direction, and the lines and planes they describe.

VECTORS AND SCALARS

More in **Vectors**

Vector

A quantity with both magnitude (size) and direction, such as displacement, velocity or force.

Scalar

A quantity with magnitude only, such as distance, speed, mass or time. Multiplying a vector by a scalar changes its length (and reverses it if the scalar is negative).

A vector

$\mathbf{a}$, $\underline{a}$, $\vec{a}$

SAY *vector a*

Three ways to show that $\boldsymbol{a}$ is a vector, not a number. Printed books use **bold**; by hand, underline it (bold is hard to write).

The vector from A to B

$\overrightarrow{AB}$

SAY *vector A B*

The displacement from point A to point B. If A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$, then $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ ("end minus start").

> EXAMPLE
> $A = (1, 2)$ and $B = (4, 6)$ give $\overrightarrow{AB} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

Column vector

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

SAY *the column vector x, y, z*

Lists a vector's components. The same vector is $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

> EXAMPLE
> $$\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}.$$

Standard unit vectors

$\mathbf{i}, \mathbf{j}, \mathbf{k}$

SAY *i, j, k*

Unit vectors in the directions of the x-, y- and z-axes. Not to be confused with the complex number $\mathbf{i}$.

Magnitude of a vector

$|\mathbf{a}|$

SAY *the magnitude of a; mod a*

The length of the vector, found by Pythagoras. At university it is often written $\|\mathbf{a}\|$ and called the norm.

> EXAMPLE
> $$\left|\begin{pmatrix} 3 \\ 4 \end{pmatrix}\right| = 5.$$

Unit vector

$\hat{\mathbf{a}}$

SAY *a hat*

The vector of length 1 in the direction of $\mathbf{a}$: $\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}$.

> EXAMPLE
> If $\mathbf{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ then $\hat{\mathbf{a}} = \begin{pmatrix} 0.6 \\ 0.8 \end{pmatrix}$.

Zero vector

The vector $\mathbf{0}$ with magnitude 0 and no direction.

Position vector

The vector from the origin to a point: A has position vector $\mathbf{a} = \overrightarrow{OA}$. Then $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$.

Resultant

The single vector equal to the sum of two or more vectors: the overall effect.

Parallel vectors

$\mathbf{a}$ and $\mathbf{b}$ are parallel when $\mathbf{b} = k\mathbf{a}$ for some scalar $k \neq 0$.

Dividing a line in a ratio

The point dividing AB in the ratio $\lambda : \mu$ has position vector $\dfrac{\mu\mathbf{a} + \lambda\mathbf{b}}{\lambda + \mu}$.

PRODUCTS OF VECTORS

More in **Vectors: lines and planes**

Scalar (dot) product

$\mathbf{a} \cdot \mathbf{b}$

SAY *a dot b*

Multiply matching components and add. The answer is a number: $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$, which is 0 exactly when the vectors are perpendicular.

> EXAMPLE
> $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = 4 + 10 + 18 = 32.$

More in **Vectors**

Angle between vectors

$\cos\theta = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$. In particular $\mathbf{a} \cdot \mathbf{b} = 0$ exactly when non-zero vectors are perpendicular.

Vector (cross) product

$\mathbf{a} \times \mathbf{b}$

SAY *a cross b*

A vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, with length $|\mathbf{a}||\mathbf{b}| \sin\theta$ (the area of the parallelogram they span). Only defined in three dimensions, and $\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$.

> EXAMPLE
> $\mathbf{i} \times \mathbf{j} = \mathbf{k}$.

Scalar triple product

$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$

$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$, the volume of the parallelepiped with edges $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ (up to sign). It is zero exactly when the three vectors are coplanar.

Coplanar

Points (or vectors) that lie in a single plane.

Linearly independent

Vectors are linearly independent if no one of them can be written as a combination of the others. Three independent vectors in 3D are not coplanar.

LINES AND PLANES

More in **Vectors: lines and planes**

Vector equation of a line

$\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$

SAY *r equals a plus lambda d*

Every point on the line, as λ runs through all real numbers: start at the point $\mathbf{a}$ and move any multiple of the direction $\mathbf{d}$.

EXAMPLE

$$\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}.$$

More in **Vectors**

Direction vector

A vector parallel to a line, the $\mathbf{d}$ in $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$.

Cartesian equation of a line in 3D

$\dfrac{x - a_1}{d_1} = \dfrac{y - a_2}{d_2} = \dfrac{z - a_3}{d_3}$ for the line through (a_1, a_2, a_3) with direction $\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$.

Skew lines

Lines in 3D that are not parallel and never meet.

Equation of a plane

$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$ (normal form), $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$ (parametric form), or $ax + by + cz = d$ (Cartesian form, with normal $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$).

Normal vector

A vector perpendicular to a plane (or to a line in 2D).

Angles with planes

The angle between two planes is the angle between their normals. The angle between a line and a plane is $90°$ minus the angle between the line and the normal.

Shortest distance

The length of the perpendicular. From (x_1, y_1, z_1) to the plane $ax + by + cz = d$ it is $\dfrac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{a^2 + b^2 + c^2}}$.

Foot of the perpendicular

The point on a line or plane closest to a given point; the line joining them meets the line or plane at right angles.

Matrices

Arrays of numbers, the transformations they describe, and their special values and vectors.

MATRICES AND THEIR ALGEBRA

More in **Matrices**

A matrix

$\mathbf{M}$

SAY *matrix M*

A rectangular array of numbers, usually a bold capital. An $m \times n$ matrix has m rows and n columns (rows first). $\mathbf{I}$ is the identity matrix and $\mathbf{0}$ (or $\mathbf{O}$) the zero matrix.

EXAMPLE
$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ is 2×3.

Order of a matrix

An $m \times n$ matrix has m rows and n columns. A matrix with as many rows as columns is **square**.

Identity matrix

$\mathbf{I}$

The square matrix $\mathbf{I}$ with 1s on the leading diagonal and 0s elsewhere. $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$.

Zero matrix

The matrix $\mathbf{0}$ with every entry 0.

Matrix multiplication

$\mathbf{AB}$ is defined when $\mathbf{A}$ has as many columns as $\mathbf{B}$ has rows; the entry in row i, column j is row i of $\mathbf{A}$ dotted with column j of $\mathbf{B}$. In general $\mathbf{AB} \neq \mathbf{BA}$.

Leading diagonal

The entries from top left to bottom right of a square matrix. Their sum is the **trace**.

Determinant

$\det \mathbf{M}$, $|\mathbf{M}|$

SAY *the determinant of M; det M*

A number that measures how a matrix scales area (or volume). The matrix has an inverse exactly when $\det \mathbf{M} \neq 0$.

EXAMPLE
$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$, so $\det \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix} = 10$.

Inverse matrix

$\mathbf{M}^{-1}$

SAY *M inverse*

The matrix with $\mathbf{M}\mathbf{M}^{-1} = \mathbf{M}^{-1}\mathbf{M} = \mathbf{I}$. It undoes the transformation $\mathbf{M}$. Note $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$, with the order reversed.

Singular and non-singular

A square matrix is singular if its determinant is 0 (it has no inverse) and non-singular otherwise.

Transpose

$\mathbf{M}^{\mathrm{T}}$

SAY *M transpose*

Swap rows and columns. Also written $\mathbf{M}'$ or $\mathbf{M}^{t}$.

EXAMPLE

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}.$$

Symmetric matrix

A square matrix equal to its transpose: $\mathbf{A}^{\mathsf{T}} = \mathbf{A}$.

Orthogonal matrix

A square matrix with $\mathbf{A}^{\mathsf{T}}\mathbf{A} = \mathbf{I}$, so its inverse is its transpose. It represents a rotation or reflection.

Consistent and inconsistent systems

A system of equations is consistent if it has at least one solution. For three planes: a single point, a line (a sheaf) or a whole plane of solutions; or none, as in a prism or parallel planes.

MATRICES AS TRANSFORMATIONS

More in **Matrices**

Linear transformation

A map T with $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ and $T(k\mathbf{v}) = kT(\mathbf{v})$. Every one is given by a matrix whose columns are the images of the basis vectors.

Rotation matrix

$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ rotates the plane anticlockwise through θ about the origin.

Reflection matrix

$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$ reflects in the line through the origin at angle θ to the x-axis.

Enlargement and stretch

$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ enlarges by scale factor k about the origin; $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ stretches by a parallel to the x-axis and b parallel to the y-axis.

Shear

A transformation that slides points parallel to a fixed line by an amount proportional to their distance from it, such as $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$. It preserves area.

Area scale factor

A 2×2 matrix multiplies areas by $|\det \mathbf{M}|$ (volumes for 3×3). A negative determinant means orientation is reversed.

Invariant point

A point mapped to itself: $\mathbf{Mx} = \mathbf{x}$.

Invariant line

A line mapped onto itself, though its points may move along it. A **line of invariant points** is one where every point stays fixed.

EIGENVALUES AND EIGENVECTORS

More in **Matrices**

Eigenvector and eigenvalue

$\mathbf{Mv} = \lambda\mathbf{v}$

A non-zero vector $\mathbf{v}$ with $\mathbf{Mv} = \lambda\mathbf{v}$: the matrix only stretches it, by the factor λ (the eigenvalue).

> EXAMPLE
> $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, eigenvalue 3.

Characteristic equation

$\det(\mathbf{M} - \lambda\mathbf{I}) = 0$, whose roots are the eigenvalues.

Diagonalisation

Writing $\mathbf{M} = \mathbf{PDP}^{-1}$ with $\mathbf{D}$ diagonal (the eigenvalues) and the columns of $\mathbf{P}$ the eigenvectors. Then $\mathbf{M}^n = \mathbf{PD}^n\mathbf{P}^{-1}$.

Complex numbers

The number system where every polynomial has its roots, and its geometry on the Argand diagram.

COMPLEX NUMBERS

More in **Complex numbers**

The imaginary unit

i

SAY *i*

A number with $\mathrm{i}^2 = -1$. Engineers write j, because i means current. Careful: $\sqrt{ab} = \sqrt{a}\sqrt{b}$ fails for negatives, so do not write $\sqrt{-1}\sqrt{-1} = \sqrt{1}$.

> EXAMPLE
> $(2 + 3\mathrm{i})(1 - \mathrm{i}) = 5 + \mathrm{i}$.

Complex number

A number $z = x + iy$ with x, y real. x is the real part and y the imaginary part; if $x = 0$ the number is (purely) imaginary.

Real and imaginary parts

$\mathrm{Re}(z)$, $\mathrm{Im}(z)$

SAY *real part of z; imaginary part of z*

For $z = a + b\mathrm{i}$, $\mathrm{Re}(z) = a$ and $\mathrm{Im}(z) = b$. The imaginary part is the real number b, not $b\mathrm{i}$.

> EXAMPLE
> $\mathrm{Im}(3 + 4\mathrm{i}) = 4$.

Complex conjugate

z^*, $\bar{z}$

SAY *z star; z bar*

Change the sign of the imaginary part. Geometrically, reflect in the real axis. Useful facts: $zz^* = |z|^2$, and real polynomials have their non-real roots in conjugate pairs.

> EXAMPLE
> $(3 + 4\mathrm{i})^* = 3 - 4\mathrm{i}$.

Complex conjugate root theorem

If a polynomial has **real** coefficients, its non-real roots come in conjugate pairs: if z is a root, so is z^*.

Fundamental theorem of algebra

Every polynomial of degree $n \geq 1$ has exactly n complex roots, counted with multiplicity.

Equal complex numbers

$a + ib = c + id$ exactly when $a = c$ and $b = d$. Comparing real and imaginary parts turns one complex equation into two real ones.

THE ARGAND DIAGRAM

More in **Complex numbers**

Argand diagram

The plane with real parts on the horizontal axis and imaginary parts on the vertical axis, so each complex number is a point.

Modulus of a complex number

$|z|$

SAY *mod z*

The distance from 0 to z on an Argand diagram: $|a+b\mathrm{i}| = \sqrt{a^2+b^2}$. And $|z-w|$ is the distance between z and w.

> EXAMPLE
> $|3+4\mathrm{i}| = 5$, and $|z-2| = 1$ is a circle of radius 1 about 2.

Argument

$\arg z$

SAY *arg z*

The angle from the positive real axis to z. The principal argument is chosen in $-\pi < \arg z \leq \pi$. Find it from a sketch, not just $\arctan\frac{b}{a}$, which gives the wrong quadrant half the time.

> EXAMPLE
> $\arg \mathrm{i} = \frac{\pi}{2}$ and $\arg(-1) = \pi$.

Principal argument

The value of the argument in $-\pi < \theta \leq \pi$.

Exponential (modulus–argument) form

$r\mathrm{e}^{\mathrm{i}\theta}$

SAY *r e to the i theta*

The complex number with modulus r and argument θ. By Euler's formula, $r\mathrm{e}^{\mathrm{i}\theta} = r(\cos\theta + \mathrm{i}\sin\theta)$, sometimes shortened to $r\,\mathrm{cis}\,\theta$.

> EXAMPLE
> $1+\mathrm{i} = \sqrt{2}\,\mathrm{e}^{\mathrm{i}\pi/4}$, and $\mathrm{e}^{\mathrm{i}\pi}+1 = 0$.

Exponential form

$re^{i\theta}$

$z = re^{i\theta}$, using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$. Multiplying multiplies moduli and adds arguments.

> EXAMPLE
> $e^{i\pi}+1 = 0$.

More in **Roots of polynomials and series**

Locus

The set of points satisfying a condition. $|z-a| = r$ is a circle centre a; $|z-a| = |z-b|$ is the perpendicular bisector of a and b; $\arg(z-a) = \theta$ is a half-line from a.

DE MOIVRE'S THEOREM AND ROOTS

More in **Roots of polynomials and series**

De Moivre's theorem

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for every integer n. Used for powers, for multiple-angle identities and for roots.

nth roots of a complex number

$z^n = w$ has exactly n solutions, equally spaced around a circle of radius $|w|^{\frac{1}{n}}$, at angles $\frac{2\pi}{n}$ apart.

A root of unity

ω

SAY *omega*

Often used for $\omega = e^{2\pi i/3}$, a non-real cube root of 1. The nth roots of unity are $1, \omega, \omega^2, \ldots, \omega^{n-1}$ with $\omega = e^{2\pi i/n}$, equally spaced round the unit circle.

> **EXAMPLE**
>
> For the cube roots: $1 + \omega + \omega^2 = 0$.

More in **Complex numbers**

Roots of unity

The n solutions of $z^n = 1$: $1, \omega, \omega^2, \ldots, \omega^{n-1}$ with $\omega = e^{\frac{2\pi i}{n}}$. They form a regular n-gon and add up to 0.

Probability

Events, how to combine them, and how to update probabilities with new information.

EXPERIMENTS AND EVENTS

More in **Probability**

Experiment and trial

An experiment is a repeatable process with uncertain results; each repetition is a trial.

Outcome

A single possible result of an experiment.

Sample space

The set of all possible outcomes, often written S or Ω.

> EXAMPLE
> Rolling a dice: $S = \{1, 2, 3, 4, 5, 6\}$.

Event

A set of outcomes, that is a subset of the sample space.

> EXAMPLE
> "An even score" is the event $\{2, 4, 6\}$.

Probability of an event

$\mathrm{P}(A)$

SAY *P of A; the probability of A*

A number from 0 (impossible) to 1 (certain). $\mathrm{P}(A \cap B)$ is the probability of both; $\mathrm{P}(A \cup B)$ of at least one.

> EXAMPLE
> $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$.

Complementary event

A'

SAY *A prime; not A*

The event A' that A does not happen. $P(A') = 1 - P(A)$.

Equally likely outcomes

When all n outcomes are equally likely, $P(A) = \frac{\text{number of outcomes in } A}{n}$.

Relative frequency

The proportion of trials in which an event happened, used as an estimate of its probability. It settles down as the number of trials grows.

Expected frequency

The number of times you would expect an event in n trials: $n \times P(A)$.

COMBINING EVENTS

More in **Probability**

Mutually exclusive

Events that cannot happen together: $P(A \cap B) = 0$, so $P(A \cup B) = P(A) + P(B)$.

Exhaustive

Events that between them cover the whole sample space, so at least one must happen.

Independent events

Events where knowing that one happened does not change the probability of the other: $P(A \cap B) = P(A)P(B)$, equivalently $P(A \mid B) = P(A)$.

Addition rule

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$ for any two events.

Conditional probability

$\mathrm{P}(A \mid B)$

SAY *P of A given B*

The probability of A once you know B has happened: $\mathrm{P}(A \mid B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$. In general $\mathrm{P}(A \mid B) \neq \mathrm{P}(B \mid A)$. If $\mathrm{P}(A \mid B) = \mathrm{P}(A)$, the events are independent.

> EXAMPLE
> Roll a fair die: $\mathrm{P}(\mathbf{6} \mid \mathbf{even}) = \frac{1}{3}$.

Conditional probability

$P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$: the probability of A once you know B has happened.

Multiplication rule

$P(A \cap B) = P(A)P(B \mid A)$: multiply along the branches of a tree diagram.

Tree diagram

A branching diagram for events in stages, with probabilities on the branches. Multiply along a path, add between paths.

Two-way table

A table of frequencies (or probabilities) classified by two features at once.

Bayes' theorem

$P(A \mid B) = \dfrac{P(B \mid A)P(A)}{P(B)}$, which reverses a conditional probability.

Statistics

Collecting and summarising data, random variables, distributions and hypothesis tests.

TYPES OF DATA

More in **Data presentation and interpretation**

Qualitative and quantitative data

Qualitative (categorical) data are descriptions, such as colour; quantitative data are numbers.

Discrete and continuous data

Discrete data take separate values (often counts); continuous data can take any value in a range (measurements).

> EXAMPLE
> Number of siblings is discrete; height is continuous.

Variable (in statistics)

A characteristic that is measured or observed and varies between individuals.

Grouped data and class boundaries

Data recorded in classes rather than as exact values. The class boundaries are the exact limits of each class, and the class width is their difference.

Bivariate data

Data with two variables recorded for each individual, such as height and mass.

AVERAGES AND SPREAD

More in **Data presentation and interpretation**

Mean

The sum of the values divided by how many there are: $\bar{x} = \frac{\sum x}{n}$, or $\frac{\sum fx}{\sum f}$ from a frequency table.

Sample mean

$\bar{x}$

SAY *x bar*

The mean of a set of data: $\bar{x} = \frac{\sum x}{n}$. As a random variable, $\bar{X}$, the mean of a random sample of size n from $\mathrm{N}(\mu, \sigma^2)$, has $\bar{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$.

> EXAMPLE
> The mean of $2, 4, 9$ is $\bar{x} = 5$.

Median

The middle value when the data are in order (the mean of the middle two if n is even). It is not affected by extreme values.

Mode

The most common value (or the modal class, for grouped data).

Range

Largest value minus smallest value: a simple, outlier-sensitive measure of spread.

Quartiles

Q_1, Q_2, Q_3

SAY *lower quartile; median; upper quartile*

The values a quarter, half and three quarters of the way through ordered data. The interquartile range is $Q_3 - Q_1$; a common rule calls a value an outlier if it is more than $1.5\times$ IQR beyond a quartile.

Interquartile range

$Q_3 - Q_1$

SAY *Q three minus Q one*

$Q_3 - Q_1$, the spread of the middle half of the data. Not affected by extreme values.

Percentile

The value below which a given percentage of the data lies. The median is the 50th percentile.

Summary statistics

S_{xx}, S_{xy}

SAY *S x x; S x y*

$S_{xx} = \sum(x - \bar{x})^2 = \sum x^2 - \frac{(\sum x)^2}{n}$ and $S_{xy} = \sum xy - \frac{\sum x \sum y}{n}$. They feed into variance, correlation and regression.

> EXAMPLE
> For $1, 2, 3$: $S_{xx} = 14 - \frac{36}{3} = 2$.

Sample standard deviation and variance

s, s^2

SAY *s; s squared*

The spread of a sample. Books differ: some divide S_{xx} by n, others by $n - 1$ (which gives an unbiased estimate of σ^2). Check which one is meant.

> EXAMPLE
> For $1, 2, 3$: dividing by $n - 1$ gives $s^2 = 1$.

Standard deviation

σ, s

SAY *sigma; s*

The square root of the variance: roughly the typical distance of a value from the mean, in the same units as the data.

Coding

Transforming data with $y = \frac{x - a}{b}$ to simplify calculations. Then $\bar{x} = a + b\bar{y}$ and $\sigma_x = b\sigma_y$: adding a constant does not change the spread.

Linear interpolation (grouped data)

Estimating the median or a quartile of grouped data by assuming the values are spread evenly through each class.

Outlier

A value that is unusually far from the rest. A common rule: more than $1.5 \times \text{IQR}$ below Q_1 or above Q_3, or more than 2 or 3 standard deviations from the mean.

Skewness

Lack of symmetry. Positive skew has a long tail to the right (typically mean > median > mode); negative skew a long tail to the left.

STATISTICAL DIAGRAMS

More in **Data presentation and interpretation**

Histogram

A diagram for continuous grouped data in which the **area** of each bar is proportional to the frequency. With unequal classes, the height is the frequency density.

Frequency density

Frequency divided by class width: the height of a histogram bar.

Box plot

A diagram showing the minimum, Q_1, median, Q_3 and maximum, with outliers often marked separately.

Cumulative frequency

The running total of the frequencies. A cumulative frequency graph plots it against the upper class boundaries and is used to estimate the median and quartiles.

Stem-and-leaf diagram

A display that keeps every value: the leading digits form the stem and the last digit is a leaf. It needs a key.

Scatter diagram

A plot of bivariate data as points, used to see whether the variables are related.

POPULATIONS AND SAMPLING

More in **Statistical sampling**

Population

The whole set of items or individuals of interest.

Sample

A selection from the population, used to draw conclusions about it.

Census

Observing or measuring every member of the population. Accurate, but expensive and impossible if testing destroys the item.

Sampling unit and sampling frame

A sampling unit is a single member of the population; the sampling frame is the list of all sampling units, from which the sample is chosen.

Simple random sampling

Every sample of size n has the same chance of being chosen, for instance by numbering the frame and using random numbers.

Systematic sampling

Choosing every kth item from an ordered list, starting from a random point among the first k.

Stratified sampling

Splitting the population into groups (strata) and sampling randomly from each in proportion to its size.

Quota sampling

An interviewer selects people to fill quotas for each group. Not random, so open to bias, but quick and needs no frame.

Opportunity sampling

Taking whoever is available at the time. Easy and cheap, but unlikely to be representative.

Cluster sampling

Dividing the population into naturally occurring groups (clusters), choosing some clusters at random, and sampling within them.

Bias

A systematic tendency for a sample or method to give results that are wrong in the same direction.

Statistic and parameter

A statistic is a quantity calculated from a sample (such as $\bar{x}$); a parameter is the corresponding value for the population (such as μ). Statistics estimate parameters.

RANDOM VARIABLES AND DISTRIBUTIONS

More in **Statistical distributions**

Random variable

A quantity whose value depends on chance, written with a capital letter such as X. Its particular values are written with lower case, x.

Probability distribution

A description of all the values a random variable can take and their probabilities. For a discrete variable, $\sum P(X = x) = 1$.

Has the distribution $X \sim$

SAY *X is distributed as; X follows*

Says which probability distribution the random variable X follows. Capital X is the random variable; lower case x is a particular value it takes.

> EXAMPLE
> $X \sim \mathrm{B}(10, 0.3)$; then $\mathrm{P}(X = x)$ is the probability that X takes the value x.

Expected value (mean) $\mathrm{E}(X)$

SAY *E of X; the expectation of X*

The long-run average value of X: $\mathrm{E}(X) = \sum x\,\mathrm{P}(X = x)$. Also written μ. Expectation is linear: $\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$.

> EXAMPLE
> For a fair die, $\mathrm{E}(X) = 3.5$.

Variance

$\mathrm{Var}(X)$

SAY *var X; the variance of X*

The average squared distance from the mean: $\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$. Also written σ^2. Note $\mathrm{Var}(aX + b) = a^2\,\mathrm{Var}(X)$.

> EXAMPLE
> For a fair die, $\mathrm{Var}(X) = \frac{91}{6} - 3.5^2 = \frac{35}{12}$.

Population mean and standard deviation

μ, σ

SAY *mu; sigma*

Greek letters describe the **population** (usually unknown); Roman letters such as $\bar{x}$ and s describe a **sample** you have measured.

More in **Data presentation and interpretation**

Cumulative distribution function

$F(x)$

$F(x) = P(X \le x)$.

Probability density and cumulative distribution

$f(x)$, $F(x)$

SAY *f of x; capital F of x*

For a continuous random variable, the area under the density $f(x)$ gives probabilities, and $F(x) = \mathrm{P}(X \le x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t$.

> EXAMPLE
> For X uniform on $[0, 2]$: $f(x) = \frac{1}{2}$ and $F(x) = \frac{x}{2}$ for $0 \le x \le 2$.

Discrete uniform distribution

Each of n values is equally likely, with probability $\frac{1}{n}$.

> EXAMPLE
> The score on a fair dice.

Binomial distribution

$\mathrm{B}(n, p)$

SAY *B n p; binomial n p*

The number of successes in n independent trials, each with probability p of success. Mean np, variance $np(1-p)$.

> EXAMPLE
> If $X \sim \mathrm{B}(10, 0.3)$ then $\mathrm{E}(X) = 3$ and $\mathrm{Var}(X) = 2.1$.

Conditions for a binomial model

A fixed number of trials n, each with two outcomes (success or failure), a constant probability of success p, and independent trials.

Poisson distribution

$\mathrm{Po}(\lambda)$

SAY *Po lambda; Poisson lambda*

The number of events in a fixed interval when they happen independently at a constant average rate λ. Its mean and variance are both λ.

> EXAMPLE
> Calls arrive at 4 per hour on average: $X \sim \mathrm{Po}(4)$, $\mathrm{P}(X = 0) = \mathrm{e}^{-4}$.

Geometric distribution

$\mathrm{Geo}(p)$

The number of trials up to and including the first success, when each trial succeeds with probability p: $P(X = r) = (1-p)^{r-1}p$.

Normal distribution

$\mathrm{N}(\mu, \sigma^2)$

SAY *N mu sigma squared; normal*

The bell-shaped distribution with mean μ and **variance** σ^2. The second number is the variance, not the standard deviation: a classic exam trap.

> EXAMPLE
> If $X \sim \mathrm{N}(50, 16)$, the standard deviation is 4.

Standard normal distribution

Z

$Z \sim N(0, 1)$. Any normal variable can be standardised with $z = \frac{x-\mu}{\sigma}$.

Standard normal and its cumulative distribution

Z, $\Phi(z)$

SAY *Z; capital phi of z ("fie")*

$Z \sim \mathrm{N}(0, 1)$, and $\Phi(z) = \mathrm{P}(Z \leq z)$. Any normal variable can be standardised: $Z = \frac{X-\mu}{\sigma}$.

> EXAMPLE
> $\Phi(1.96) \approx 0.975$ and $\Phi(-z) = 1 - \Phi(z)$.

Continuity correction

Adjusting by 0.5 when a continuous distribution approximates a discrete one, such as $P(X \leq 10) \approx P(Y < 10.5)$.

Normal approximation to the binomial

If n is large and p is close to $\frac{1}{2}$, $B(n, p) \approx N(np, np(1-p))$, with a continuity correction.

Central limit theorem

For a large sample from any distribution with mean μ and variance σ^2, the sample mean is approximately $N\left(\mu, \frac{\sigma^2}{n}\right)$.

CORRELATION AND REGRESSION

More in **Data presentation and interpretation**

Correlation

A linear relationship between two variables: positive if they tend to increase together, negative if one tends to decrease as the other increases.

Correlation coefficient

r, ρ

SAY *r; rho ("row")*

The product moment correlation coefficient, $r = \frac{S_{xy}}{\sqrt{S_{xx}S_{yy}}}$, measures linear correlation in a sample, with $-1 \leq r \leq 1$.

Its population version is ρ.

> EXAMPLE
> $r = 0.95$ means strong positive linear correlation, not that one variable causes the other.

More in **Hypothesis testing**

Correlation and causation

Correlation does not imply causation: two variables can be correlated because of a third factor, or by chance.

Explanatory and response variables

The explanatory (independent) variable is the one set or measured first; the response (dependent) variable is the one that may depend on it.

Regression line

The line of best fit $y = a + bx$ found by least squares; it is used to predict y from x. b is the change in y for each unit increase in x.

Interpolation and extrapolation

Interpolation predicts within the range of the data and is usually reliable; extrapolation predicts outside it and may not be.

HYPOTHESIS TESTING

More in **Hypothesis testing**

Hypothesis test

A procedure for deciding, from sample data, whether there is enough evidence to reject a claim about a population parameter.

Null and alternative hypotheses

H_0, H_1

SAY *H nought; H one*

H_0 is the default claim you assume is true; H_1 is what you look for evidence of. Both are statements about a population parameter, never about the sample.

> EXAMPLE
> $H_0 : p = 0.3$, $H_1 : p > 0.3$ is a one-tailed test.

One-tailed and two-tailed tests

A one-tailed test has H_1 in one direction ($p > 0.3$); a two-tailed test has $H_1 : p \neq 0.3$, and the significance level is split between the two tails.

Test statistic

The value calculated from the sample that is used to decide the outcome of the test.

Significance level

α

SAY *alpha*

The probability of rejecting H_0 when it is actually true, fixed before the test (often 5%). The critical region is chosen so that this probability is at most α.

Critical region

The set of values of the test statistic that lead to rejecting H_0. Its boundary is the **critical value**; the rest is the acceptance region.

p-value

The probability, assuming H_0 is true, of a result at least as extreme as the one observed. Reject H_0 if it is less than the significance level.

Actual significance level

For a discrete distribution, the true probability of the critical region under H_0, which is usually a little below the nominal level.

Type I and Type II errors

A Type I error rejects H_0 when it is true (probability = the significance level); a Type II error fails to reject H_0 when it is false.

Conclusion in context

The last line of a test, stated in words about the situation and without certainty: "there is (in)sufficient evidence to suggest …".

Mechanics

The language of motion, forces, moments, energy and the models used to describe them.

MODELS AND UNITS

More in **Kinematics**

Mathematical model

A simplified mathematical description of a real situation, built on assumptions, used to make predictions and then refined.

Particle

An object treated as a single point: its size is ignored, so it cannot rotate and air resistance can be neglected.

Rigid body and lamina

A rigid body has size but does not bend or deform. A lamina is a flat object whose thickness is ignored.

Light

A light object (such as a string or rod) is modelled as having zero mass, so the tension is the same throughout it.

Inextensible

A string that does not stretch, so connected objects move with the same speed and acceleration.

Smooth and rough

A smooth surface or pulley has no friction; a rough one does.

Uniform

A uniform rod or lamina has its mass spread evenly, so its centre of mass is at its geometric centre.

Rod and beam

A rod is a long, thin rigid body whose width is ignored; a beam is a rod that may bend in reality but is modelled as rigid.

Units with negative powers

$\mathrm{m\,s^{-2}}$

SAY *metres per second squared*

A negative power means "per". So $\mathrm{m\,s^{-1}}$ is metres per second, and a moment is measured in newton metres, $\mathrm{N\,m}$.

> EXAMPLE
> $1\ \mathrm{N} = 1\ \mathrm{kg\,m\,s^{-2}}$.

More in **Forces and Newton's laws**

Newton

N

The SI unit of force: the force giving a mass of 1 kg an acceleration of $1\ \mathrm{m\,s^{-2}}$, so $1\ \mathrm{N} = 1\ \mathrm{kg\,m\,s^{-2}}$.

KINEMATICS

More in **Kinematics**

Displacement and distance

Displacement is the vector position relative to a starting point (it can be negative); distance is the total length travelled (never negative).

Velocity and speed

Velocity is the rate of change of displacement, a vector; speed is its magnitude.

Acceleration

The rate of change of velocity. Deceleration (retardation) is acceleration opposite to the direction of motion.

The suvat letters

s, u, v, a, t

SAY *suvat*

Displacement, initial velocity, final velocity, acceleration and time, for motion with **constant** acceleration. Five equations link them, each missing one letter.

> EXAMPLE
> $v^2 = u^2 + 2as$ has no t.

Velocity–time graph

The gradient is the acceleration and the area under it is the displacement. On a displacement–time graph the gradient is the velocity.

Variable acceleration

When acceleration is not constant: $v = \frac{\mathrm{d}s}{\mathrm{d}t}$, $a = \frac{\mathrm{d}v}{\mathrm{d}t}$, and integrate to go back, using initial conditions for the constants.

Projectile

An object moving freely under gravity alone. Its horizontal velocity is constant and its vertical acceleration is g downwards, so its path is a parabola.

Time of flight, range and maximum height

For a projectile launched from ground level: the time until it lands, the horizontal distance travelled by then, and the greatest height reached (where the vertical velocity is zero).

FORCES AND NEWTON'S LAWS

More in **Forces and Newton's laws**

Force

A push or pull, measured in newtons. A vector, so it has a direction.

Common force letters

R, T, F, W

SAY *reaction; tension; friction; weight*

Usual names in force diagrams: R (or N) the normal reaction, T tension (or thrust), F friction, W weight.

Acceleration due to gravity

g

SAY *g*

About $9.8\ \mathrm{m\,s^{-2}}$ near the Earth's surface. A body of mass m has weight $W = mg$.

> EXAMPLE
> A $5\ \mathrm{kg}$ mass weighs $49\ \mathrm{N}$.

Weight and mass

Mass (kg) is the amount of matter; weight is the gravitational force on it, $W = mg$ newtons, acting vertically downwards.

Normal reaction

The force from a surface on an object, perpendicular to the surface.

Tension and thrust

Tension is the pulling force in a string or rod; thrust (compression) is the pushing force in a rod.

Friction

A force along a rough surface opposing the motion (or the tendency to move). It satisfies $F \le \mu R$, with equality when sliding or about to slide.

Coefficient of friction

μ

SAY *mu ("myoo")*

Friction satisfies $F \le \mu R$, with equality when the body is moving or on the point of moving (limiting friction).

> EXAMPLE
> On a rough plane with $\mu = 0.4$ and $R = 20\ \mathrm{N}$, friction can reach at most $8\ \mathrm{N}$.

Limiting equilibrium

An object on the point of moving: friction has its maximum value $F = \mu R$.

Equilibrium

An object is in equilibrium when the resultant force on it is zero (and, for a rigid body, the resultant moment too). It is at rest or moving at constant velocity.

Resultant force

The single force with the same effect as all the forces acting together: their vector sum.

Resolving

Splitting a force into components in two perpendicular directions: a force F at angle θ to a direction has component $F\cos\theta$ along it and $F\sin\theta$ perpendicular to it.

Newton's first law

An object stays at rest or moves with constant velocity unless a resultant force acts on it.

Newton's second law

$F = ma$

Resultant force = mass × acceleration: $F = ma$.

Newton's third law

If A exerts a force on B, then B exerts an equal and opposite force on A.

Connected particles

Particles joined by a light inextensible string (often over a smooth pulley), so they share the same acceleration and the tension is the same throughout the string.

MOMENTS AND CENTRES OF MASS

More in **Moments**

Moment

The turning effect of a force about a point: force × perpendicular distance from the point to the line of action. Units N m.

Pivot (fulcrum)

The point about which a body turns or could turn.

Equilibrium of a rigid body

The resultant force is zero and the total moment about **every** point is zero: clockwise moments equal anticlockwise moments.

On the point of tilting

A body about to turn about a support: the reaction at any other support has just become zero.

Couple

Two equal and opposite forces whose lines of action are different. It has a turning effect but no resultant force.

Centre of mass

The point where the whole mass of a body can be treated as acting. For particles, $\bar{x} = \dfrac{\sum m_i x_i}{\sum m_i}$.

MOMENTUM, WORK AND ENERGY

More in **Forces and Newton's laws**

Momentum

$m\mathbf{v}$

Mass × velocity, $m\boldsymbol{v}$: a vector, measured in N s (or kg m s^{-1}). In a collision the total momentum is conserved.

Impulse

The change in momentum, $I = mv - mu$; for a constant force, force × time.

Coefficient of restitution

e

$e = \dfrac{\text{speed of separation}}{\text{speed of approach}}$ in a collision, with $0 \le e \le 1$. $e = 1$ is perfectly elastic and $e = 0$ perfectly inelastic.

Work done

Force × distance moved in the direction of the force, measured in joules (J).

Kinetic and potential energy

Kinetic energy is $\frac{1}{2}mv^2$; gravitational potential energy is mgh relative to a chosen level. Without friction or other resistances, their total is conserved.

Power

The rate of doing work, measured in watts (W). For a vehicle, power = driving force × speed.

Hooke's law

The tension in an elastic string or spring is $T = \frac{\lambda x}{l}$, where λ is the modulus of elasticity, x the extension and l the natural length. The elastic potential energy is $\frac{\lambda x^2}{2l}$.

Greek letters

How to say them, and what they usually mean.

THE GREEK ALPHABET

Alpha

α

SAY *"al-fa"*

Angles; significance level; roots of polynomials (α, β, γ).

More in **Roots of polynomials and series**

Beta

β

SAY *"bee-ta"*

Second angle or second root.

Gamma

γ, Γ

SAY *"gam-a"*

Third root; Euler–Mascheroni constant $\gamma \approx 0.577$; capital Γ is the gamma function, which extends factorials: $\Gamma(n) = (n-1)!$.

Delta

δ, Δ

SAY *"del-ta"*

A small change (δx); a total change (Δ); the discriminant; the δ in ε–δ definitions.

Epsilon

ε

SAY *"ep-si-lon"*

A small positive number, especially in limits: "for every $\varepsilon > 0$ there is a $\delta > 0$ …". Also errors.

Zeta

ζ

SAY *"zee-ta"*

The Riemann zeta function, $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, with $\zeta(2) = \frac{\pi^2}{6}$.

Eta

η

SAY *"ee-ta"*

Efficiency in mechanics; a spare variable.

Theta

θ

SAY *"thee-ta"*

The most common letter for an angle; the angle in polar coordinates (r, θ).

More in **Trigonometry**

Iota

ι

SAY *"eye-oh-ta"*

Rarely used in school mathematics; it is the letter behind the word "jot".

Kappa

κ

SAY *"cap-a"*

Curvature of a curve; some physical constants.

Lambda

λ

SAY *"lam-da"*

Parameter of a line; eigenvalues; the Poisson rate.

More in **Vectors**

Mu

μ

SAY *"myoo"*

Population mean; coefficient of friction; second line parameter.

Nu

ν

SAY *"nyoo"*

Degrees of freedom in statistics. Easy to confuse with v.

Xi

ξ

SAY *"ksy" or "zai"*

The universal set in British books; a spare variable.

Omicron

o, O

SAY *"oh-mi-cron"*

Almost never used, because it looks exactly like the letter o.

Pi

π, Π

SAY *"pie"*

The circle constant π; capital Π for products.

Rho

ρ

SAY *"row"*

Population correlation; density.

Sigma

σ, Σ

SAY *"sig-ma"*

Standard deviation σ; capital Σ for sums.

Tau

τ

SAY *"tor" or "tow"*

Torque; time constants. Some argue for $\tau = 2\pi$ as the "true" circle constant.

Upsilon

υ, Υ

SAY *"oop-si-lon"*

Rarely used; capital Υ appears occasionally in physics.

Phi

ϕ, Φ

SAY *"fie"*

Angles; the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$; capital Φ for the standard normal distribution function.

Chi

χ

SAY *"kie"*

The χ^2 (chi-squared) test and distribution.

Psi

ψ

SAY *"sigh" or "psy"*

Angles; the wavefunction in quantum mechanics.

Omega

ω, Ω

SAY *"oh-meg-a"*

Angular speed; roots of unity; capital Ω for a sample space (or ohms).

Spotted a mistake, or a definition that should be here? Email lpotry31@gmail.com. More at lemmamathematics.com.