DEFINITIONS

# Calculus

*Differentiation, integration, differential equations and series expansions.*

**56** definitions **5** subtopics By Lucas Potrykus

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HOW TO USE THIS

Each definition gives the exact meaning of a word or symbol. Where there is a symbol, it is shown next to the name, with how to say it out loud. Most definitions have a short example.

Learn definitions word for word: in proofs and in exams, the precise wording is what earns the marks.

Searchable online, with links to every topic, at lemmamathematics.com/definitions.

## DIFFERENTIATION

More in **Differentiation**

### Limit

$\lim_{x \to a} f(x) = L$ means $f(x)$ can be made as close as you like to $L$ by taking $x$ close enough to $a$ (but not equal to it).

### Derivative

The gradient of a curve at a point, and the instantaneous rate of change of $y$ with respect to $x$: $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$.

### Differentiation from first principles

Finding a derivative directly from the limit definition, by expanding $f(x+h) - f(x)$, dividing by $h$ and letting $h \to 0$.

### Derivative (Leibniz notation)

$\frac{\mathrm{d}y}{\mathrm{d}x}$

SAY *dee y by dee x*

The rate of change of $y$ with respect to $x$: the gradient of the curve. It is not really a fraction, but it often behaves like one, as in the chain rule $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$.

EXAMPLE

If $y = x^3$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2$.

## Derivative (Lagrange notation)

$f'(x)$, $f''(x)$

SAY *f dash x; f double dash x (America: "f prime")*

The derivative of the function $f$, then the second derivative. For higher derivatives write $f^{(n)}(x)$, with the brackets showing it is not a power.

> EXAMPLE
> If $f(x) = \sin x$ then $f'(x) = \cos x$ and $f''(x) = -\sin x$.

## Second derivative

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$

SAY *dee two y by dee x squared*

The derivative of the derivative: how fast the gradient is changing. Its sign tells you whether a curve is concave up or down. Note where the 2s go.

> EXAMPLE
> If $y = x^3$ then $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6x$.

## The nth derivative

$f^{(n)}(x)$

SAY *the nth derivative of f*

Differentiate $f$ a total of $n$ times. Used in Maclaurin and Taylor series.

> EXAMPLE
> If $f(x) = \mathrm{e}^{2x}$ then $f^{(n)}(x) = 2^n\mathrm{e}^{2x}$.

More in **Further calculus**

## Time derivatives (Newton notation)

$\dot{x}$, $\ddot{x}$

SAY *x dot; x double dot*

A dot means differentiate with respect to **time**. Common in mechanics and differential equations.

> EXAMPLE
> If $x$ is position, $\dot{x}$ is velocity and $\ddot{x}$ is acceleration.

More in **Kinematics**

## A small change in x

$\delta x$

SAY *delta x*

A small (but non-zero) increase in $x$. Derivatives arise as $\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$. Some books write $h$ instead.

## Change in

$\Delta$

SAY *delta*

The total change in a quantity: final value minus initial value. Unlike $\delta$, it need not be small.

> EXAMPLE
> If speed rises from $u$ to $v$, then $\Delta v = v - u$.

More in **Kinematics**

## Differentiable

A function is differentiable at a point if its derivative exists there: the graph has a single, non-vertical tangent. A corner (like $|x|$ at 0) is not differentiable.

## Tangent

The straight line that touches a curve at a point and has the same gradient as the curve there.

### Normal

The line through a point of a curve perpendicular to the tangent there. Its gradient is $-\frac{1}{f'(x)}$.

### Chain rule

For a composite function: $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}y}{\mathrm{d}u} \times \dfrac{\mathrm{d}u}{\mathrm{d}x}$.

> EXAMPLE
> $\frac{\mathrm{d}}{\mathrm{d}x}(3x+1)^5 = 15(3x+1)^4$.

### Product rule

$\dfrac{\mathrm{d}}{\mathrm{d}x}(uv) = u\dfrac{\mathrm{d}v}{\mathrm{d}x} + v\dfrac{\mathrm{d}u}{\mathrm{d}x}$.

### Quotient rule

$\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{u}{v}\right) = \dfrac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$.

### Implicit differentiation

Differentiating an equation in $x$ and $y$ term by term without first making $y$ the subject, using the chain rule: $\frac{\mathrm{d}}{\mathrm{d}x}\big(g(y)\big) = g'(y)\frac{\mathrm{d}y}{\mathrm{d}x}$.

> EXAMPLE
> $x^2 + y^2 = 25$ gives $2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.

### Connected rates of change

Linking the rates of change of related quantities with the chain rule.

> EXAMPLE
> For a sphere, $\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}t} = 4\pi r^2\frac{\mathrm{d}r}{\mathrm{d}t}$.

## STATIONARY POINTS AND THE SHAPE OF A CURVE

More in **Differentiation**

### Stationary point

A point where $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$: the tangent is horizontal. It is a local maximum, a local minimum or a stationary point of inflection.

### Turning point

A stationary point where the gradient changes sign: a local maximum or a local minimum.

### Local maximum and minimum

A local maximum is higher than all nearby points of the curve; a local minimum is lower. The **global** maximum or minimum is the highest or lowest value over the whole domain.

### Second derivative test

At a stationary point, $f''(x) > 0$ means a minimum and $f''(x) < 0$ a maximum. If $f''(x) = 0$ the test fails: check the gradient on either side.

### Concave and convex

A curve is convex (concave up) where $f''(x) > 0$, bending like a cup, and concave (concave down) where $f''(x) < 0$.

## Point of inflection

A point where the curve changes between concave and convex, so $f''(x)$ changes sign. It need not be stationary.

> EXAMPLE
> $y = x^3$ has a stationary point of inflection at the origin.

## Optimisation

Finding the largest or smallest value of a quantity, usually by writing it as a function of one variable and finding its stationary points.

# INTEGRATION

More in **Integration techniques**

## Integral (antiderivative)

A function whose derivative is the given function. Integration is the reverse of differentiation.

## Indefinite integral

$\int f(x)\,\mathrm{d}x$

SAY *the integral of f of x with respect to x*

The family of all functions whose derivative is $f(x)$, so you add a constant $+c$. The $\mathrm{d}x$ says which variable you are integrating with respect to; it is not decoration.

> EXAMPLE
> $\int 2x\,\mathrm{d}x = x^2 + c$.

## Constant of integration

$+c$

The $+c$ in an indefinite integral: every antiderivative of a function differs from any other by a constant.

## Definite integral

$\int_a^b f(x)\,\mathrm{d}x$

SAY *the integral from a to b of f of x dee x*

A number: the signed area between the curve and the $x$-axis from $x = a$ to $x = b$ (area below the axis counts as negative).

> EXAMPLE
> $\int_0^1 x^2\,\mathrm{d}x = \frac{1}{3}$.

## Evaluate between limits

$\left[F(x)\right]_a^b$

SAY *F of x between a and b*

Shorthand for $F(b) - F(a)$, used in the middle of working out a definite integral.

> EXAMPLE
> $\left[x^2\right]_1^3 = 9 - 1 = 8$.

## Fundamental theorem of calculus

If $F' = f$ then $\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$: the area under a curve can be found from any antiderivative.

## Area under a curve

$\int_a^b f(x)\,\mathrm{d}x$ gives the area between $y = f(x)$ and the $x$-axis when $f(x) \geq 0$. Regions below the axis count as negative, so find them separately.

### Area between curves

$\int_a^b \big(f(x) - g(x)\big)\,\mathrm{d}x$ where $f(x) \geq g(x)$ on $[a, b]$ (top curve minus bottom curve).

### Reverse chain rule (integration by inspection)

Recognising an integrand as the result of the chain rule, such as $\int f'(x)\big(f(x)\big)^n\,\mathrm{d}x = \frac{(f(x))^{n+1}}{n+1} + c$ and $\int \frac{f'(x)}{f(x)}\,\mathrm{d}x = \ln|f(x)| + c$.

### Integration by substitution

Changing the variable, $u = g(x)$, to turn the integral into a simpler one. Remember to replace $\mathrm{d}x$ and, for a definite integral, the limits.

### Integration by parts

$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$, the reverse of the product rule.

> EXAMPLE
> $\int xe^x\,\mathrm{d}x = xe^x - e^x + c$.

### Volume of revolution

Rotating $y = f(x)$ from $x = a$ to $x = b$ through $2\pi$ about the $x$-axis gives volume $\pi\int_a^b y^2\,\mathrm{d}x$.

More in **Further calculus**

### Mean value of a function

The average height of $f$ over $[a, b]$: $\dfrac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$.

More in **Further calculus**

### Improper integral

An integral with an infinite limit, or an integrand that is infinite somewhere in the range. It is defined as a limit, and converges if that limit exists.

> EXAMPLE
> $\int_1^\infty \frac{1}{x^2}\,\mathrm{d}x = \lim_{b\to\infty}\left(1 - \frac{1}{b}\right) = 1$.

More in **Further calculus**

## DIFFERENTIAL EQUATIONS

More in **Differential equations**

### Differential equation

An equation involving derivatives of an unknown function. Its **order** is that of the highest derivative.

### General solution

The solution with all its arbitrary constants (one for each order), giving the whole family of solution curves.

### Particular solution

The one solution that also satisfies given conditions, such as an initial value $y(0) = 2$, which fix the constants.

### Separable differential equation

One that can be written $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y)$, and solved by $\int \frac{1}{g(y)}\,\mathrm{d}y = \int f(x)\,\mathrm{d}x$.

More in **Integration techniques**

### Linear differential equation

One in which $y$ and its derivatives appear only to the first power and are not multiplied together, such as $\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$.

### Integrating factor

For $\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$, multiplying through by $e^{\int P\,\mathrm{d}x}$ turns the left side into $\frac{\mathrm{d}}{\mathrm{d}x}\left(ye^{\int P\,\mathrm{d}x}\right)$.

### Homogeneous linear equation

A linear differential equation with zero on the right-hand side, such as $ay'' + by' + cy = 0$.

### Auxiliary equation

For $ay'' + by' + cy = 0$, the quadratic $am^2 + bm + c = 0$. Its roots decide the form of the solution: $Ae^{m_1 x} + Be^{m_2 x}$, $(A + Bx)e^{mx}$, or $e^{px}(A\cos qx + B\sin qx)$.

### Complementary function

The general solution of the homogeneous equation. The full solution is complementary function plus particular integral.

### Particular integral

Any one solution of the full (non-homogeneous) equation, usually found by trying a function like the right-hand side.

### Simple harmonic motion

Motion with acceleration proportional to displacement and directed towards a fixed point: $\ddot{x} = -\omega^2 x$, with solution $x = a\sin(\omega t + \varepsilon)$ and period $\frac{2\pi}{\omega}$.

### Damping

A resistance that makes oscillations die away. Light damping still oscillates, critical damping returns fastest without oscillating, heavy damping returns slowly without oscillating.

### Coupled differential equations

Two or more differential equations in which each unknown's rate depends on the others, such as predator–prey models.

## SERIES EXPANSIONS

More in **Further calculus**

### Maclaurin series

$f(x) = f(0) + f'(0)x + \dfrac{f''(0)}{2!}x^2 + \dfrac{f'''(0)}{3!}x^3 + \cdots$, a power series that matches $f$ and all its derivatives at $x = 0$.

> EXAMPLE
> $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ for all $x$.

### Taylor series

The same idea centred on $x = a$: $f(x) = f(a) + f'(a)(x - a) + \dfrac{f''(a)}{2!}(x - a)^2 + \cdots$

### Power series

A series of the form $a_0 + a_1 x + a_2 x^2 + \cdots$, which may converge only for some values of $x$.

## L'Hôpital's rule

If $f(a) = g(a) = 0$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$, when the second limit exists.

EXAMPLE

$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1$.

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Spotted a mistake, or a definition that should be here? Email lpotry31@gmail.com. More at lemmamathematics.com.