DEFINITIONS

Complex numbers

The number system where every polynomial has its roots, and its geometry on the Argand diagram.

18 definitions **3** subtopics By Lucas Potrykus

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HOW TO USE THIS

Each definition gives the exact meaning of a word or symbol. Where there is a symbol, it is shown next to the name, with how to say it out loud. Most definitions have a short example.

Learn definitions word for word: in proofs and in exams, the precise wording is what earns the marks.

Searchable online, with links to every topic, at lemmamathematics.com/definitions.

COMPLEX NUMBERS

More in **Complex numbers**

The imaginary unit

i

SAY *i*

A number with $\mathrm{i}^2 = -1$. Engineers write j, because i means current. Careful: $\sqrt{ab} = \sqrt{a}\sqrt{b}$ fails for negatives, so do not write $\sqrt{-1}\sqrt{-1} = \sqrt{1}$.

EXAMPLE

$(2 + 3\mathrm{i})(1 - \mathrm{i}) = 5 + \mathrm{i}$.

Complex number

A number $z = x + iy$ with x, y real. x is the real part and y the imaginary part; if $x = 0$ the number is (purely) imaginary.

Real and imaginary parts

$\mathrm{Re}(z)$, $\mathrm{Im}(z)$

SAY *real part of z; imaginary part of z*

For $z = a + b\mathrm{i}$, $\mathrm{Re}(z) = a$ and $\mathrm{Im}(z) = b$. The imaginary part is the real number b, not $b\mathrm{i}$.

EXAMPLE

$\mathrm{Im}(3 + 4\mathrm{i}) = 4$.

Complex conjugate

z^*, $\bar{z}$

SAY *z star; z bar*

Change the sign of the imaginary part. Geometrically, reflect in the real axis. Useful facts: $zz^* = |z|^2$, and real polynomials have their non-real roots in conjugate pairs.

EXAMPLE

$(3 + 4\mathrm{i})^* = 3 - 4\mathrm{i}$.

Complex conjugate root theorem

If a polynomial has **real** coefficients, its non-real roots come in conjugate pairs: if z is a root, so is z^*.

Fundamental theorem of algebra

Every polynomial of degree $n \geq 1$ has exactly n complex roots, counted with multiplicity.

Equal complex numbers

$a + ib = c + id$ exactly when $a = c$ and $b = d$. Comparing real and imaginary parts turns one complex equation into two real ones.

THE ARGAND DIAGRAM

More in **Complex numbers**

Argand diagram

The plane with real parts on the horizontal axis and imaginary parts on the vertical axis, so each complex number is a point.

Modulus of a complex number

$|z|$

SAY *mod z*

The distance from 0 to z on an Argand diagram: $|a + b\mathrm{i}| = \sqrt{a^2 + b^2}$. And $|z - w|$ is the distance between z and w.

> EXAMPLE
> $|3 + 4\mathrm{i}| = 5$, and $|z - 2| = 1$ is a circle of radius 1 about 2.

Argument

$\arg z$

SAY *arg z*

The angle from the positive real axis to z. The principal argument is chosen in $-\pi < \arg z \leq \pi$. Find it from a sketch, not just $\arctan \frac{b}{a}$, which gives the wrong quadrant half the time.

> EXAMPLE
> $\arg \mathrm{i} = \frac{\pi}{2}$ and $\arg(-1) = \pi$.

Principal argument

The value of the argument in $-\pi < \theta \leq \pi$.

Exponential (modulus–argument) form

$re^{\mathrm{i}\theta}$

SAY *r e to the i theta*

The complex number with modulus r and argument θ. By Euler's formula, $re^{\mathrm{i}\theta} = r(\cos\theta + \mathrm{i}\sin\theta)$, sometimes shortened to $r\,\mathrm{cis}\,\theta$.

> EXAMPLE
> $1 + \mathrm{i} = \sqrt{2}\,e^{\mathrm{i}\pi/4}$, and $e^{\mathrm{i}\pi} + 1 = 0$.

Exponential form

$re^{i\theta}$

$z = re^{i\theta}$, using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$. Multiplying multiplies moduli and adds arguments.

> EXAMPLE
> $e^{i\pi} + 1 = 0$.

More in **Roots of polynomials and series**

Locus

The set of points satisfying a condition. $|z-a|=r$ is a circle centre a; $|z-a|=|z-b|$ is the perpendicular bisector of a and b; $\arg(z-a)=\theta$ is a half-line from a.

DE MOIVRE'S THEOREM AND ROOTS

More in **Roots of polynomials and series**

De Moivre's theorem

$(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta$ for every integer n. Used for powers, for multiple-angle identities and for roots.

nth roots of a complex number

$z^n=w$ has exactly n solutions, equally spaced around a circle of radius $|w|^{\frac{1}{n}}$, at angles $\frac{2\pi}{n}$ apart.

A root of unity

ω

SAY *omega*

Often used for $\omega=\mathrm{e}^{2\pi\mathrm{i}/3}$, a non-real cube root of 1. The nth roots of unity are $1,\omega,\omega^2,\ldots,\omega^{n-1}$ with $\omega=\mathrm{e}^{2\pi\mathrm{i}/n}$, equally spaced round the unit circle.

> EXAMPLE
> For the cube roots: $1+\omega+\omega^2=0$.

More in **Complex numbers**

Roots of unity

The n solutions of $z^n=1$: $1,\omega,\omega^2,\ldots,\omega^{n-1}$ with $\omega=e^{\frac{2\pi i}{n}}$. They form a regular n-gon and add up to 0.

Spotted a mistake, or a definition that should be here? Email lpotry31@gmail.com. More at lemmamathematics.com.