DEFINITIONS

Functions and graphs

What a function is, the words for describing one, and how graphs are transformed.

36 definitions **3** subtopics By Lucas Potrykus

CONTENTS

1. Functions 18
2. Graphs and transformations 9
3. Exponentials and logarithms 9

HOW TO USE THIS

Each definition gives the exact meaning of a word or symbol. Where there is a symbol, it is shown next to the name, with how to say it out loud. Most definitions have a short example.

Learn definitions word for word: in proofs and in exams, the precise wording is what earns the marks.

Searchable online, with links to every topic, at lemmamathematics.com/definitions.

FUNCTIONS

More in **Functions and graphs**

A function from A to B

$f : A \to B$

SAY *f from A to B*

f assigns to every element of the domain A exactly one element of the codomain B. The range (or image) is the set of outputs actually reached, which can be smaller than B.

EXAMPLE

$f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ has codomain $\mathbb{R}$ but range $[0, \infty)$.

Maps to

$x \mapsto f(x)$

SAY *x maps to …*

Describes what a function does to a single input, without naming the function.

EXAMPLE

$x \mapsto 2x + 1$ doubles and adds one, so $3 \mapsto 7$.

Mapping

A rule that sends each input to one or more outputs. A function is a mapping that sends each input to **exactly one** output.

Domain

The set of inputs a function is defined on. It is part of the function: changing the domain gives a different function.

EXAMPLE

$f(x) = \sqrt{x - 3}$ needs the domain $x \geq 3$.

Range

The set of outputs a function actually takes, as x runs over the domain. It is also called the image.

> EXAMPLE
> $f(x) = x^2 + 1$, $x \in \mathbb{R}$, has range $f(x) \geq 1$.

Codomain

The set the outputs are declared to lie in, written after the arrow in $f : A \to B$. The range is the part of the codomain actually reached.

One-to-one function

A function in which different inputs always give different outputs. Only one-to-one functions have inverses. (At university: injective.)

> EXAMPLE
> $f(x) = x^3$ is one-to-one; $f(x) = x^2$ on $\mathbb{R}$ is not, since $f(-2) = f(2)$.

Many-to-one function

A function in which some output comes from more than one input.

Composite function

$fg(x)$, $(f \circ g)(x)$

SAY *f of g of x; f after g*

$f(g(x))$: apply g first, then f. A level writes $fg(x)$; university writes $(f \circ g)(x)$. The order matters.

> EXAMPLE
> With $f(x) = x^2$ and $g(x) = x + 1$: $fg(x) = (x + 1)^2$ but $gf(x) = x^2 + 1$.

Inverse function

f^{-1}

SAY *f inverse*

The function that undoes f: $f^{-1}(f(x)) = x$. It is **not** $\frac{1}{f(x)}$. Only one-to-one functions have inverses; its graph is the reflection of $y = f(x)$ in $y = x$.

> EXAMPLE
> If $f(x) = 2x + 1$ then $f^{-1}(x) = \frac{x-1}{2}$.

Self-inverse function

A function that is its own inverse: $f(f(x)) = x$. Its graph is symmetric in $y = x$.

> EXAMPLE
> $f(x) = \frac{1}{x}$ and $f(x) = 5 - x$.

Even function

A function with $f(-x) = f(x)$ for every x. Its graph is symmetric in the y-axis.

> EXAMPLE
> x^2, $\cos x$, $|x|$.

Odd function

A function with $f(-x) = -f(x)$ for every x. Its graph has rotational symmetry of order 2 about the origin.

> EXAMPLE
> x^3, $\sin x$, $\tan x$.

Periodic function

A function with $f(x+T)=f(x)$ for every x, for some $T>0$. The smallest such T is the period.

> EXAMPLE
> $\sin x$ has period 2π; $\tan x$ has period π.

Increasing and decreasing functions

f is increasing on an interval if $x_1<x_2$ implies $f(x_1)\le f(x_2)$ (strictly increasing with $<$). For a differentiable function, $f'(x)>0$ throughout means strictly increasing.

Piecewise function

A function defined by different rules on different parts of its domain.

> EXAMPLE
> $$f(x)=\begin{cases}x^2 & x<0\\ 2x & x\ge 0\end{cases}$$

Modulus (absolute value)

$|x|$

SAY *mod x; the absolute value of x*

The size of x ignoring its sign: $|x|=x$ if $x\ge 0$ and $-x$ if $x<0$. Think of $|a-b|$ as the distance between a and b.

> EXAMPLE
> $|-3|=3$, and $|x-2|<1 \iff 1<x<3$.

Floor and ceiling

$\lfloor x\rfloor$, $\lceil x\rceil$

SAY *floor of x; ceiling of x*

$\lfloor x\rfloor$ is the largest integer $\le x$ (round down); $\lceil x\rceil$ is the smallest integer $\ge x$ (round up). Watch negative numbers.

> EXAMPLE
> $\lfloor 2.7\rfloor=2$, $\lfloor -2.7\rfloor=-3$ and $\lceil 2.1\rceil=3$.

GRAPHS AND TRANSFORMATIONS

More in **Functions and graphs**

Intercept

Where a graph crosses an axis. The y-intercept is at $x=0$; the x-intercepts are the real roots.

Asymptote

A line that a curve approaches more and more closely, without necessarily ever reaching it. It can be vertical, horizontal or oblique (slanted).

> EXAMPLE
> $y=\frac{1}{x-2}$ has asymptotes $x=2$ and $y=0$.

Translation

A shift without turning or stretching. $y=f(x)+a$ moves the graph up by a; $y=f(x+a)$ moves it **left** by a. As a vector: $\begin{pmatrix}-a\\0\end{pmatrix}$.

Stretch

$y=af(x)$ stretches the graph vertically by scale factor a. $y=f(ax)$ stretches it horizontally by scale factor $\frac{1}{a}$.

Reflection of a graph

$y = -f(x)$ reflects the graph in the x-axis; $y = f(-x)$ reflects it in the y-axis; the inverse function's graph is the reflection in $y = x$.

Modulus graphs

$y = |f(x)|$ reflects the parts of the graph below the x-axis upwards. $y = f(|x|)$ keeps the part for $x \geq 0$ and reflects it in the y-axis.

Reciprocal graph

The graph of $y = \frac{1}{f(x)}$. Roots of f become vertical asymptotes, and large values become small ones.

Sketch

A graph drawn by hand showing the correct shape, the intercepts, turning points and asymptotes, with their coordinates or equations labelled. It does not need to be to scale.

Gradient of a line

$m = \dfrac{y_2 - y_1}{x_2 - x_1}$, the change in y for each unit increase in x.

More in **Straight lines and circles**

EXPONENTIALS AND LOGARITHMS

More in **Exponentials and logarithms**

Exponential function

A function $f(x) = a^x$ with base $a > 0$, $a \neq 1$. Its graph passes through $(0, 1)$ and has the x-axis as an asymptote.

Euler's number

e

SAY *e*

The constant $\mathrm{e} \approx 2.71828$, chosen so that $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x = \mathrm{e}^x$. $\exp(x)$ is another way to write e^x, handy when the power is long.

> EXAMPLE
> $\mathrm{e} = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$.

Exponential growth and decay

A quantity grows exponentially when its rate of increase is proportional to its size: $N = N_0 e^{kt}$ with $k > 0$. With $k < 0$ it decays.

Half-life

The time taken for an exponentially decaying quantity to halve. For $N = N_0 e^{-kt}$ it is $\frac{\ln 2}{k}$.

Logarithm

$\log_a x$ is the power you raise a to in order to get x: $\log_a x = y \iff a^y = x$.

> EXAMPLE
> $\log_2 32 = 5$ since $2^5 = 32$.

Logarithm to base a

$\log_a x$

SAY *log to base a of x*

The power you raise a to in order to get x. Plain $\log x$ is ambiguous: base 10 on calculators and in school, but usually base e at university.

> EXAMPLE
> $\log_2 8 = 3$ because $2^3 = 8$.

Natural logarithm

$\ln x$

SAY *"log x" or "lin x"*

The logarithm to base e: $\ln x = y \iff \mathrm{e}^y = x$. Defined for $x > 0$ only.

> EXAMPLE
> $\ln 1 = 0$, $\ln \mathrm{e} = 1$ and $\mathrm{e}^{\ln x} = x$.

Laws of logarithms

$\log_a(xy) = \log_a x + \log_a y$, $\log_a \frac{x}{y} = \log_a x - \log_a y$, $\log_a x^k = k \log_a x$, and the change of base rule $\log_a x = \frac{\log_b x}{\log_b a}$.

Reducing to linear form

Taking logs turns $y = ax^n$ into $\log y = n \log x + \log a$, a straight line in $(\log x, \log y)$, and $y = kb^x$ into $\log y = x \log b + \log k$, a straight line in $(x, \log y)$.

Spotted a mistake, or a definition that should be here? Email lpotry31@gmail.com. More at lemmamathematics.com.