DEFINITIONS

Matrices

Arrays of numbers, the transformations they describe, and their special values and vectors.

24 definitions **3** subtopics By Lucas Potrykus

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HOW TO USE THIS

Each definition gives the exact meaning of a word or symbol. Where there is a symbol, it is shown next to the name, with how to say it out loud. Most definitions have a short example.

Learn definitions word for word: in proofs and in exams, the precise wording is what earns the marks.

Searchable online, with links to every topic, at lemmamathematics.com/definitions.

MATRICES AND THEIR ALGEBRA

More in **Matrices**

A matrix

M

SAY *matrix M*

A rectangular array of numbers, usually a bold capital. An $m \times n$ matrix has m rows and n columns (rows first). $\mathbf{I}$ is the identity matrix and $\mathbf{0}$ (or $\mathbf{O}$) the zero matrix.

EXAMPLE

$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ is 2×3.

Order of a matrix

An $m \times n$ matrix has m rows and n columns. A matrix with as many rows as columns is **square**.

Identity matrix

I

The square matrix $\mathbf{I}$ with 1s on the leading diagonal and 0s elsewhere. $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$.

Zero matrix

The matrix $\mathbf{0}$ with every entry 0.

Matrix multiplication

$\mathbf{AB}$ is defined when $\mathbf{A}$ has as many columns as $\mathbf{B}$ has rows; the entry in row i, column j is row i of $\mathbf{A}$ dotted with column j of $\mathbf{B}$. In general $\mathbf{AB} \neq \mathbf{BA}$.

Leading diagonal

The entries from top left to bottom right of a square matrix. Their sum is the **trace**.

Determinant

$\det \mathbf{M}$, $|\mathbf{M}|$

SAY *the determinant of M; det M*

A number that measures how a matrix scales area (or volume). The matrix has an inverse exactly when $\det \mathbf{M} \neq 0$.

EXAMPLE

$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$, so $\det \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix} = 10$.

Inverse matrix

$\mathbf{M}^{-1}$

SAY *M inverse*

The matrix with $\mathbf{M}\mathbf{M}^{-1} = \mathbf{M}^{-1}\mathbf{M} = \mathbf{I}$. It undoes the transformation $\mathbf{M}$. Note $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$, with the order reversed.

Singular and non-singular

A square matrix is singular if its determinant is 0 (it has no inverse) and non-singular otherwise.

Transpose

$\mathbf{M}^{\mathrm{T}}$

SAY *M transpose*

Swap rows and columns. Also written $\mathbf{M}'$ or $\mathbf{M}^t$.

EXAMPLE

$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$.

Symmetric matrix

A square matrix equal to its transpose: $\mathbf{A}^{\mathsf{T}} = \mathbf{A}$.

Orthogonal matrix

A square matrix with $\mathbf{A}^{\mathsf{T}}\mathbf{A} = \mathbf{I}$, so its inverse is its transpose. It represents a rotation or reflection.

Consistent and inconsistent systems

A system of equations is consistent if it has at least one solution. For three planes: a single point, a line (a sheaf) or a whole plane of solutions; or none, as in a prism or parallel planes.

MATRICES AS TRANSFORMATIONS

More in **Matrices**

Linear transformation

A map T with $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ and $T(k\mathbf{v}) = kT(\mathbf{v})$. Every one is given by a matrix whose columns are the images of the basis vectors.

Rotation matrix

$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ rotates the plane anticlockwise through θ about the origin.

Reflection matrix

$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$ reflects in the line through the origin at angle θ to the x-axis.

Enlargement and stretch

$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ enlarges by scale factor k about the origin; $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ stretches by a parallel to the x-axis and b parallel to the y-axis.

Shear

A transformation that slides points parallel to a fixed line by an amount proportional to their distance from it, such as $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$. It preserves area.

Area scale factor

A 2×2 matrix multiplies areas by $|\det \mathbf{M}|$ (volumes for 3×3). A negative determinant means orientation is reversed.

Invariant point

A point mapped to itself: $\mathbf{Mx} = \mathbf{x}$.

Invariant line

A line mapped onto itself, though its points may move along it. A **line of invariant points** is one where every point stays fixed.

EIGENVALUES AND EIGENVECTORS

More in **Matrices**

Eigenvector and eigenvalue

$\mathbf{Mv} = \lambda\mathbf{v}$

A non-zero vector $\mathbf{v}$ with $\mathbf{Mv} = \lambda\mathbf{v}$: the matrix only stretches it, by the factor λ (the eigenvalue).

> EXAMPLE
> $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, eigenvalue 3.

Characteristic equation

$\det(\mathbf{M} - \lambda\mathbf{I}) = 0$, whose roots are the eigenvalues.

Diagonalisation

Writing $\mathbf{M} = \mathbf{PDP}^{-1}$ with $\mathbf{D}$ diagonal (the eigenvalues) and the columns of $\mathbf{P}$ the eigenvectors. Then $\mathbf{M}^n = \mathbf{PD}^n\mathbf{P}^{-1}$.

Spotted a mistake, or a definition that should be here? Email lpotry31@gmail.com. More at lemmamathematics.com.