DEFINITIONS

Sequences and series

Lists of numbers, their sums, what happens in the long run, and proof by induction.

26 definitions **3** subtopics By Lucas Potrykus

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HOW TO USE THIS

Each definition gives the exact meaning of a word or symbol. Where there is a symbol, it is shown next to the name, with how to say it out loud. Most definitions have a short example.

Learn definitions word for word: in proofs and in exams, the precise wording is what earns the marks.

Searchable online, with links to every topic, at lemmamathematics.com/definitions.

SEQUENCES

More in **Sequences and series**

Sequence

An ordered list of numbers $u_1, u_2, u_3, \dots$ (its terms). It can be finite or infinite.

The nth term of a sequence

u_n

SAY *u n; u sub n*

The term in position n. The whole sequence is written (u_n) or $\{u_n\}$. A recurrence relation such as $u_{n+1} = 2u_n + 1$ defines each term from the previous one.

EXAMPLE

If $u_n = 3n - 1$, then $u_1 = 2$, $u_2 = 5$, $u_{10} = 29$.

And so on (ellipsis)

$\dots$, $\cdots$

SAY *dot dot dot; and so on*

The pattern continues. Low dots $\dots$ go between commas; centred dots $\cdots$ go between operations such as $+$ and $\times$.

EXAMPLE

$1, 2, \dots, n$ and $1 + 2 + \cdots + n$.

Recurrence relation

A rule giving each term from the one (or ones) before, together with a starting value.

EXAMPLE

$u_{n+1} = 2u_n + 1$, $u_1 = 1$ gives $1, 3, 7, 15, \dots$

Arithmetic sequence

A sequence with a constant difference d between consecutive terms: $u_n = a + (n-1)d$.

> EXAMPLE
> $5, 8, 11, 14, \ldots$ has $a = 5$, $d = 3$.

Geometric sequence

A sequence with a constant ratio r between consecutive terms: $u_n = ar^{n-1}$.

> EXAMPLE
> $3, 6, 12, 24, \ldots$ has $a = 3$, $r = 2$.

Increasing and decreasing sequences

A sequence is increasing if $u_{n+1} > u_n$ for every n and decreasing if $u_{n+1} < u_n$ for every n. Either is called **monotonic**.

Periodic sequence

A sequence whose terms repeat in a cycle. The length of the cycle is the order (or period).

> EXAMPLE
> $u_{n+1} = \frac{1}{1-u_n}$, $u_1 = 2$ gives $2, -1, \frac{1}{2}, 2, \ldots$, period 3.

Convergent and divergent sequences

A sequence converges if its terms get as close as you like to a fixed number L (its limit) and stay there. Otherwise it diverges, for instance by growing without bound or oscillating.

> EXAMPLE
> $\frac{1}{n} \to 0$ converges; $(-1)^n$ and n^2 diverge.

Tends to (or maps to a set)

$\to$

SAY *tends to; approaches*

In limits, $x \to a$ means x gets arbitrarily close to a. In function notation, $f : A \to B$ means f takes inputs from A and gives outputs in B.

> EXAMPLE
> As $x \to \infty$, $\frac{1}{x} \to 0$.

More in **Differentiation**

Limit

$\lim_{n\to\infty}$

SAY *the limit as n tends to infinity of …*

The value a sequence or function gets arbitrarily close to. It can exist even when the value at that point does not, as in $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

> EXAMPLE
> $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \mathrm{e}$.

Fibonacci sequence

$1, 1, 2, 3, 5, 8, \ldots$, defined by $F_{n+2} = F_{n+1} + F_n$. The ratio of consecutive terms tends to the golden ratio $\frac{1+\sqrt{5}}{2}$.

SERIES

More in **Sequences and series**

Series

The sum of the terms of a sequence.

Sum (sigma notation)

$\sum_{r=1}^{n} a_r$

SAY *the sum from r equals 1 to n of a r*

Add up a_r for every whole number r from the bottom limit to the top limit. The letter r is a dummy variable: using k instead changes nothing.

> EXAMPLE
> $\sum_{r=1}^{4} r^2 = 1 + 4 + 9 + 16 = 30$.

Product (pi notation)

$\prod_{r=1}^{n} a_r$

SAY *the product from r equals 1 to n of a r*

Multiply a_r together for every r from the bottom limit to the top. An empty product is 1, just as an empty sum is 0.

> EXAMPLE
> $\prod_{r=1}^{4} r = 24 = 4!$.

More in **Roots of polynomials and series**

Partial sum and sum to infinity

S_n, S_∞

SAY *S n; S infinity*

S_n is the sum of the first n terms. S_∞ is the limit of S_n as $n \to \infty$, when that limit exists.

> EXAMPLE
> For $1 + \frac{1}{2} + \frac{1}{4} + \cdots$: $S_n = 2 - \frac{1}{2^{n-1}}$ and $S_\infty = 2$.

Arithmetic series

The sum of an arithmetic sequence: $S_n = \frac{n}{2}\big(2a + (n-1)d\big) = \frac{n}{2}(a + l)$ where l is the last term.

> EXAMPLE
> $1 + 2 + \cdots + 100 = 50 \times 101 = 5050$.

Geometric series

The sum of a geometric sequence: $S_n = \dfrac{a(1 - r^n)}{1 - r}$ for $r \neq 1$.

Sum to infinity

S_∞

SAY *S infinity*

The limit of the partial sums S_n as $n \to \infty$, if it exists. For a geometric series it exists exactly when $|r| < 1$, and then $S_\infty = \dfrac{a}{1 - r}$.

> EXAMPLE
> $1 + \frac{1}{2} + \frac{1}{4} + \cdots = 2$.

Convergent series

A series whose partial sums tend to a finite limit. Terms tending to 0 is necessary but not sufficient.

> EXAMPLE
> $\sum \frac{1}{n}$ diverges (the harmonic series) although $\frac{1}{n} \to 0$.

Standard sums

$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$, $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$, $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$.

More in **Roots of polynomials and series**

Method of differences

Writing each term of a series as $f(r) - f(r+1)$, so that almost everything cancels in the sum (it **telescopes**).

> EXAMPLE
> $\sum_{r=1}^{n} \frac{1}{r(r+1)} = \sum \left(\frac{1}{r} - \frac{1}{r+1}\right) = 1 - \frac{1}{n+1}$.

More in **Roots of polynomials and series**

PROOF BY INDUCTION

More in **Mathematical induction**

Base case

The first step of an induction proof: checking the statement for the smallest value (often $n = 1$).

Inductive hypothesis

The assumption, in an induction proof, that the statement is true for $n = k$.

Inductive step

Proving that if the statement is true for $n = k$, then it is true for $n = k + 1$.

Conclusion of an induction proof

The closing sentence: since the statement is true for $n = 1$, and true for $n = k + 1$ whenever it is true for $n = k$, it is true for all $n \geq 1$. Exams give a mark for it.

Spotted a mistake, or a definition that should be here? Email lpotry31@gmail.com. More at lemmamathematics.com.