DEFINITIONS

# Trigonometry

*Angles, the trigonometric and hyperbolic functions, and the identities that connect them.*

**29** definitions **4** subtopics By Lucas Potrykus

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HOW TO USE THIS

Each definition gives the exact meaning of a word or symbol. Where there is a symbol, it is shown next to the name, with how to say it out loud. Most definitions have a short example.

Learn definitions word for word: in proofs and in exams, the precise wording is what earns the marks.

Searchable online, with links to every topic, at lemmamathematics.com/definitions.

## ANGLES AND MEASURES

More in **Trigonometry**

### Radian

The angle at the centre of a circle subtended by an arc equal in length to the radius. $\pi$ radians $= 180°$, so 1 radian is about $57.3°$.

### Arc length

The length of part of a circle's circumference: $s = r\theta$ with $\theta$ in radians.

### Sector and segment

A sector is the region between two radii and an arc, area $\frac{1}{2}r^2\theta$. A segment is the region between a chord and an arc, area $\frac{1}{2}r^2(\theta - \sin\theta)$.

### Unit circle

The circle of radius 1 centred at the origin. The point at angle $\theta$ from the positive $x$-axis is $(\cos\theta, \sin\theta)$, which defines $\sin$ and $\cos$ for every angle.

### CAST diagram

A memory aid for the signs of the trig functions in the four quadrants: All positive in the first, Sine in the second, Tangent in the third, Cosine in the fourth.

### Exact values

The values of sin, cos and tan at $0$, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$ and $\frac{\pi}{2}$ that can be written with surds.

> EXAMPLE
> $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, $\tan\frac{\pi}{4} = 1$.

### Small-angle approximations

For small $\theta$ in radians: $\sin\theta \approx \theta$, $\tan\theta \approx \theta$, $\cos\theta \approx 1 - \frac{\theta^2}{2}$.

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### Bearing

A direction given as an angle measured clockwise from north, written with three figures.

> EXAMPLE
> South-west is $225°$.

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### Angle of elevation and depression

The angle above the horizontal (elevation) or below it (depression) of the line of sight to an object.

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## TRIGONOMETRIC FUNCTIONS

More in **Trigonometry**

### Sine, cosine and tangent

$\sin$, $\cos$, $\tan$

In a right-angled triangle: $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$, $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$, $\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sin\theta}{\cos\theta}$.

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### Sine rule

In any triangle, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$. With two angles and a side, or two sides and a non-included angle (which can give two triangles: the **ambiguous case**).

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### Cosine rule

In any triangle, $a^2 = b^2 + c^2 - 2bc\cos A$. Use it with three sides, or two sides and the angle between them.

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### Area of a triangle

$\text{Area} = \frac{1}{2}ab\sin C$, where $C$ is the angle between sides $a$ and $b$.

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### Powers of trig functions (a trap)

$\sin^2 x$, $\sin^{-1} x$

SAY *sine squared x; inverse sine of x*

A clash of conventions: $\sin^2 x$ means $(\sin x)^2$, but $\sin^{-1} x$ means the **inverse** function $\arcsin x$, not $\frac{1}{\sin x}$ (that is $\operatorname{cosec} x$).

> EXAMPLE
> $\sin^2\frac{\pi}{6} = \frac{1}{4}$, while $\sin^{-1}\frac{1}{2} = \frac{\pi}{6}$.

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### Reciprocal trig functions

$\sec$, $\operatorname{cosec}$, $\cot$

SAY *secant; cosecant; cotangent*

$\sec x = \frac{1}{\cos x}$, $\operatorname{cosec} x = \frac{1}{\sin x}$ (written $\csc x$ in America) and $\cot x = \frac{1}{\tan x}$. A useful memory aid: the third letter of each name tells you which function it is the reciprocal of.

> EXAMPLE
> $1 + \tan^2 x \equiv \sec^2 x$.

More in **Trigonometric identities and equations**

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## Inverse sine

$\arcsin x$

SAY *arc sine of x*

The angle in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ whose sine is $x$. Similarly $\arccos x \in [0, \pi]$ and $\arctan x \in (-\frac{\pi}{2}, \frac{\pi}{2})$. These restricted ranges make them functions.

> EXAMPLE
> $\arcsin \frac{1}{2} = \frac{\pi}{6}$, even though $\sin \frac{5\pi}{6}$ is also $\frac{1}{2}$.

## Principal value

The single value an inverse trig function returns. **arcsin** and **arctan** give values in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and **arccos** in $[0, \pi]$; other solutions of an equation are found from these using symmetry and the period.

## Amplitude

The largest distance of a wave from its mean level. $y = a \sin x$ has amplitude $|a|$.

# TRIGONOMETRIC IDENTITIES

More in **Trigonometric identities and equations**

## Pythagorean identities

$\sin^2 \theta + \cos^2 \theta \equiv 1$, and dividing through, $1 + \tan^2 \theta \equiv \sec^2 \theta$ and $1 + \cot^2 \theta \equiv \operatorname{cosec}^2 \theta$.

## Compound angle formulae

$\sin(A \pm B) \equiv \sin A \cos B \pm \cos A \sin B$, $\cos(A \pm B) \equiv \cos A \cos B \mp \sin A \sin B$ and $\tan(A \pm B) \equiv \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$.

## Double angle formulae

$\sin 2A \equiv 2 \sin A \cos A$, $\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2 \cos^2 A - 1 \equiv 1 - 2 \sin^2 A$ and $\tan 2A \equiv \dfrac{2 \tan A}{1 - \tan^2 A}$.

## Harmonic form (R cos form)

Writing $a \cos \theta + b \sin \theta$ as $R \cos(\theta - \alpha)$ (or a similar form), with $R = \sqrt{a^2 + b^2}$ and $\tan \alpha = \frac{b}{a}$. It makes the maximum, minimum and solutions easy to read off.

> EXAMPLE
> $3 \cos \theta + 4 \sin \theta = 5 \cos(\theta - 0.927\ldots)$, so its maximum is 5.

## The t-formulae

With $t = \tan \frac{\theta}{2}$: $\sin \theta = \dfrac{2t}{1 + t^2}$, $\cos \theta = \dfrac{1 - t^2}{1 + t^2}$, $\tan \theta = \dfrac{2t}{1 - t^2}$.

## Factor formulae

Sums of sines and cosines as products, such as $\sin P + \sin Q \equiv 2 \sin \frac{P+Q}{2} \cos \frac{P-Q}{2}$.

# HYPERBOLIC FUNCTIONS

More in **Hyperbolic functions**

## Hyperbolic functions

$\sinh$, $\cosh$, $\tanh$

SAY *"shine" or "sinch"; "cosh"; "than" or "tanch"*

$\sinh x = \frac{e^x - e^{-x}}{2}$, $\cosh x = \frac{e^x + e^{-x}}{2}$, $\tanh x = \frac{\sinh x}{\cosh x}$. They play the role for the hyperbola $x^2 - y^2 = 1$ that $\sin$ and $\cos$ play for the circle.

> EXAMPLE
> $\cosh^2 x - \sinh^2 x \equiv 1$.

## Inverse hyperbolic functions

$\operatorname{arsinh} x$

SAY *"ar-shine" x*

Inverses of the hyperbolic functions, also written $\sinh^{-1} x$. They can be written with logarithms. The prefix "ar" stands for *area*, not *arc*.

> EXAMPLE
> $\operatorname{arsinh} x = \ln\left(x + \sqrt{x^2 + 1}\right)$.

## Hyperbolic identities

$\cosh^2 x - \sinh^2 x \equiv 1$, $\sinh 2x \equiv 2 \sinh x \cosh x$ and $\cosh 2x \equiv \cosh^2 x + \sinh^2 x$.

## Osborn's rule

A trig identity becomes the matching hyperbolic identity if you replace each trig function by its hyperbolic version and change the sign of every product (or implied product) of two sines.

> EXAMPLE
> $\cos^2 x + \sin^2 x \equiv 1$ becomes $\cosh^2 x - \sinh^2 x \equiv 1$.

## Inverse hyperbolic functions as logarithms

$\operatorname{arsinh} x = \ln\left(x + \sqrt{x^2 + 1}\right)$, $\operatorname{arcosh} x = \ln\left(x + \sqrt{x^2 - 1}\right)$ for $x \geq 1$, $\operatorname{artanh} x = \frac{1}{2} \ln \frac{1+x}{1-x}$ for $|x| < 1$.

Spotted a mistake, or a definition that should be here? Email lpotry31@gmail.com. More at lemmamathematics.com.