Pure Mathematics

Paper 1 · Foundation: model solutions

LP1/P · Level 1

How to use these solutions. Try each question under timed conditions first, then mark it with the mark scheme. Read the full solution for every question, even the ones you got right: they often show a faster method, a check, or a common error to avoid. Each solution begins with *the idea*, then works through numbered steps; remarks marked ↪ explain *why*. Each ends with the boxed answer and how the marks are earned.

Question 1 *Indices and surds* 6 marks

In this question you must show all stages of your working. Solutions relying entirely on calculator technology are not acceptable.

(a) *Simplify fully* $(2x^3)^2 \times 3x^{-1}$. *(2)*

(b) *Write* $\sqrt{48} + \sqrt{27}$ *in the form* $a\sqrt{3}$, *where* a *is an integer.* *(2)*

(c) *Find the exact value of* $8^{-\frac{2}{3}}$. *(2)*

Solution.

The idea.

Three separate warm-up skills: the laws of indices, simplifying surds by pulling out square factors, and reading a fractional negative power as "reciprocal, root, power".

Step 1. (a) Square the bracket.

A power outside a bracket applies to **everything** inside it, both the number and the power of x:

$$(2x^3)^2 = 2^2 \times (x^3)^2 = 4x^{3\times 2} = 4x^6.$$

↪ *The common slip is* $(2x^3)^2 = 2x^6$, *squaring the* x^3 *but forgetting the* 2.

Step 2. (a) Multiply the two parts.

Now multiply by $3x^{-1}$. Multiply the numbers, and **add** the powers of x (same base, so the powers add):

$$4x^6 \times 3x^{-1} = (4 \times 3)\, x^{6+(-1)} = 12x^5.$$

Step 3. (b) Look for square factors.

To simplify a surd, split the number into (largest square factor) $\times$ (what's left):

$$48 = 16 \times 3, \qquad 27 = 9 \times 3.$$

Then use $\sqrt{ab} = \sqrt{a}\sqrt{b}$:

$$\sqrt{48} = \sqrt{16}\sqrt{3} = 4\sqrt{3}, \qquad \sqrt{27} = \sqrt{9}\sqrt{3} = 3\sqrt{3}.$$

↪ *If you'd used* $48 = 4 \times 12$*, you'd get* $2\sqrt{12}$*, which still simplifies. Always hunt for the* ***largest*** *square factor.*

Step 4. (b) Collect like surds.

Both terms are now multiples of $\sqrt{3}$, so they add like $4y + 3y$:

$$4\sqrt{3} + 3\sqrt{3} = 7\sqrt{3}, \qquad a = 7.$$

↪ *Careful:* $\sqrt{48} + \sqrt{27} \neq \sqrt{75}$*. Square roots do not add like that.*

Step 5. (c) Read the power in three pieces.

The power $-\frac{2}{3}$ contains three instructions:

- the **minus** sign means "take the reciprocal";
- the **denominator 3** means "cube root";
- the **numerator 2** means "square".

Do the root first, because it keeps the numbers small:

$$8^{\frac{1}{3}} = 2 \quad (\text{since } 2^3 = 8), \qquad 8^{\frac{2}{3}} = 2^2 = 4.$$

Step 6. (c) Apply the reciprocal.

$$8^{-\frac{2}{3}} = \frac{1}{8^{\frac{2}{3}}} = \frac{1}{4}.$$

↪ *A negative power never makes the answer negative. It makes it a reciprocal.*

Answer. (a) $12x^5$ (b) $7\sqrt{3}$ (c) $\dfrac{1}{4}$

Common errors.

- $(2x^3)^2 = 2x^6$ (forgetting to square the 2).
- Multiplying the powers, $x^6 \times x^{-1} = x^{-6}$, instead of adding them.
- Writing $8^{-\frac{2}{3}} = -4$.

How the marks are earned.

M1 **(a)** $(2x^3)^2 = 4x^6$, or a correct use of a power law on the whole expression *(Allow 2^2x^6.)*

A1 $12x^5$ *(cao)*

M1 **(b)** $\sqrt{48} = 4\sqrt{3}$ or $\sqrt{27} = 3\sqrt{3}$ *(One surd correctly simplified.)*

A1 $7\sqrt{3}$ *(cao)*

M1 **(c)** $8^{\frac{1}{3}} = 2$ or $8^{\frac{2}{3}} = 4$ or $8^2 = 64$ seen, or the negative power treated as a reciprocal

A1 $\frac{1}{4}$ *(Accept 0.25.)*

Question 2 *Simultaneous equations* 7 marks

> *Solve the simultaneous equations*
>
> $$y = x + 1, \qquad x^2 + y^2 = 13.$$

Solution.

The idea.

One equation is linear and one is quadratic, so substitute the linear one into the quadratic. That leaves a quadratic in one letter, which can have up to two solutions, so expect ***two pairs*** *of answers.*

Step 1. Choose what to substitute.

The first equation already says what y is, so replace every y in the second equation with $x + 1$:

$$x^2 + (x + 1)^2 = 13.$$

↪ *Always substitute the* ***linear*** *equation into the quadratic one, never the other way round.*

Step 2. Expand the bracket carefully.

$(x + 1)^2$ means $(x + 1)(x + 1)$, which has **three** terms:

$$(x + 1)^2 = x^2 + 2x + 1.$$

So the equation becomes

$$x^2 + x^2 + 2x + 1 = 13 \quad \Longrightarrow \quad 2x^2 + 2x + 1 - 13 = 0 \quad \Longrightarrow \quad 2x^2 + 2x - 12 = 0.$$

Step 3. Simplify by a common factor.

Every term is even, so divide through by 2. This doesn't change the solutions and makes factorising easier:

$$x^2 + x - 6 = 0.$$

Step 4. Factorise and solve.

I need two numbers that multiply to -6 and add to $+1$: these are $+3$ and -2.

$$(x + 3)(x - 2) = 0 \quad \Longrightarrow \quad x = -3 \quad \text{or} \quad x = 2.$$

Step 5. Find the matching y values.

Use the **linear** equation $y = x + 1$, because it gives exactly one y for each x:

- $x = 2$: $y = 2 + 1 = 3$;
- $x = -3$: $y = -3 + 1 = -2$.

↪ *If you used $x^2 + y^2 = 13$ instead, each x would give two y values ($\pm$), and you'd have to throw half of them away.*

Step 6. Check both pairs.

- $(2, 3)$: $2^2 + 3^2 = 4 + 9 = 13$ ✓
- $(-3, -2)$: $9 + 4 = 13$ ✓

Geometrically, these are the two points where the line $y = x + 1$ crosses the circle of radius $\sqrt{13}$ about the origin.

Answer. $x = 2,\ y = 3$ and $x = -3,\ y = -2$

Common errors.

- Expanding $(x + 1)^2$ as $x^2 + 1$.
- Giving four answers by using $y = \pm\sqrt{13 - x^2}$.
- Not pairing the values: "$x = 2, -3$ and $y = 3, -2$" loses the last mark.

How the marks are earned.

M1 Substitutes $y = x + 1$ into $x^2 + y^2 = 13$ (or $x = y - 1$) *(Must give an equation in one variable.)*
A1 $x^2 + x - 6 = 0$ (or $2x^2 + 2x - 12 = 0$) *(Any correct three-term quadratic $= 0$.)*
M1 Solves their three-term quadratic *(Factorising, formula or completing the square.)*
A1 $x = 2$ and $x = -3$
M1 Substitutes at least one x value into $y = x + 1$
A1 $(2, 3)$ *(Must be paired correctly.)*
A1 $(-3, -2)$

∎

Question 3 *Straight lines* 6 marks

The points A and B have coordinates $(1, 3)$ and $(5, 11)$ respectively.

(a) *Find an equation of the line AB, giving your answer in the form $y = mx + c$.* *(3)*

(b) *Find an equation of the perpendicular bisector of AB, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers.* *(3)*

Solution.

The idea.

Everything about straight lines comes from two facts: the gradient is $\dfrac{\text{change in } y}{\text{change in } x}$, and perpendicular gradients multiply to -1. The area in (d) is easy once you notice MC is a height of the triangle.

Step 1. (a) Gradient of AB.

$$m_{AB} = \frac{y_B - y_A}{x_B - x_A} = \frac{11 - 3}{5 - 1} = \frac{8}{4} = 2.$$

↪ *Keep the order consistent: if B is first on top, B must be first underneath.*

Step 2. (a) Equation of the line.

Use $y - y_1 = m(x - x_1)$ with the point $A(1, 3)$:

$$y - 3 = 2(x - 1) \quad \Longrightarrow \quad y = 2x - 2 + 3 \quad \Longrightarrow \quad y = 2x + 1.$$

Check with B: $2 \times 5 + 1 = 11$ ✓. Checking the other point costs five seconds and catches most errors.

Step 3. (b) What a perpendicular bisector is.

The perpendicular bisector of AB is the line that

1. passes through the **midpoint** of AB (bisector), and
2. meets AB at **right angles** (perpendicular).

So I need a point (the midpoint) and a gradient (perpendicular to 2).

Step 4. (b) The midpoint.

Average the coordinates:

$$M = \left(\frac{1 + 5}{2}, \frac{3 + 11}{2}\right) = (3, 7).$$

Step 5. (b) The perpendicular gradient.

If two lines are perpendicular, their gradients multiply to -1. So the new gradient is the **negative reciprocal** of 2:

$$m_\perp = -\frac{1}{2}, \qquad \text{check: } 2 \times \left(-\tfrac{1}{2}\right) = -1. \checkmark$$

Step 6. (b) The equation in the required form.

$$y - 7 = -\frac{1}{2}(x - 3).$$

Multiply both sides by 2 to clear the fraction:

$$2y - 14 = -(x - 3) = -x + 3.$$

Bring everything to one side:

$$x + 2y - 17 = 0.$$

Check with M: $3 + 14 - 17 = 0$ ✓.

> **Answer.** (a) $y = 2x + 1$ (b) $x + 2y - 17 = 0$

Common errors.

- Using the reciprocal $\frac{1}{2}$ instead of the **negative** reciprocal $-\frac{1}{2}$.
- Using A or B instead of the midpoint in (b).
- Leaving fractions in the final form when the question asks for integers.

How the marks are earned.

M1 **(a)** Gradient $= \dfrac{11 - 3}{5 - 1} = 2$ *(Correct method for the gradient.)*
M1 Uses $y - 3 = m(x - 1)$ or $y = mx + c$ with a point *(With their gradient.)*
A1 $y = 2x + 1$
B1 **(b)** Midpoint $(3, 7)$
M1 Uses gradient $-\dfrac{1}{2}$ through their midpoint *(Negative reciprocal of their (a) gradient.)*
A1 $x + 2y - 17 = 0$ *(Or any integer multiple.)*

∎

Question 4 *Coordinate geometry* 4 marks

The points A and B have coordinates $(1,3)$ and $(5,11)$. The perpendicular bisector of AB has equation $x+2y-17=0$, and it crosses the x-axis at the point C.

(a) *Find the coordinates of C.* ***(1)***

(b) *Find the area of triangle ABC.* ***(3)***

Solution.

The idea.

*Find C from the equation of the bisector. Then notice that, because C lies on the perpendicular bisector of AB, the line from the midpoint M of AB to C meets AB at right angles: it is a ready-made **height** of the triangle.*

Step 1. (a) Crossing the x-axis.

On the x-axis, $y=0$:

$$x+0-17=0 \quad\Longrightarrow\quad x=17, \qquad C=(17,0).$$

Step 2. (b) Spot the height.

C is on the perpendicular bisector, so the line MC is **perpendicular** to AB. That makes MC the height of triangle ABC if I take AB as the base:

$$\text{Area}=\frac{1}{2}\times AB\times MC.$$

↪ *Because C is on the perpendicular bisector, $CA=CB$: the triangle is isosceles, with M at the foot of its line of symmetry.*

Step 3. (b) The two lengths.

Using Pythagoras, $\text{distance}=\sqrt{(\Delta x)^2+(\Delta y)^2}$:

$$AB=\sqrt{(5-1)^2+(11-3)^2}=\sqrt{16+64}=\sqrt{80},$$

$$MC=\sqrt{(17-3)^2+(0-7)^2}=\sqrt{196+49}=\sqrt{245}.$$

Step 4. (b) The area.

$$\text{Area}=\frac{1}{2}\sqrt{80}\sqrt{245}=\frac{1}{2}\sqrt{80\times 245}=\frac{1}{2}\sqrt{19600}=\frac{1}{2}\times 140=70.$$

Check with the shoelace formula for vertices $(1,3),(5,11),(17,0)$:

$$\tfrac{1}{2}\,|1(11-0)+5(0-3)+17(3-11)|=\tfrac{1}{2}\,|11-15-136|=\tfrac{1}{2}\times 140=70.\checkmark$$

Answer. (a) $C(17,0)$ (b) 70

Another method.

For (b), you could also enclose the triangle in the rectangle $1 \le x \le 17$, $0 \le y \le 11$ (area 176) and subtract the three right-angled triangles around it: $\frac{1}{2}(4)(8) + \frac{1}{2}(12)(11) + \frac{1}{2}(16)(3) = 16 + 66 + 24 = 106$, leaving $176 - 106 = 70$.

How the marks are earned.

B1ft **(a)** $(17, 0)$ *(Follow through from their (b).)*

M1 **(b)** Correct method for a relevant length, e.g. $AB = \sqrt{80}$ and $MC = \sqrt{245}$; or a correct start to the shoelace formula

M1 $\frac{1}{2} \times \sqrt{80} \times \sqrt{245}$, or a complete correct method

A1 70

∎

Question 5 *Quadratics* 9 marks

$$f(x) = x^2 - 6x + 5$$

(a) *Write $f(x)$ in the form $(x+p)^2 + q$, where p and q are integers.* *(2)*

(b) *Hence state the coordinates of the turning point of the curve $y = f(x)$.* *(1)*

(c) *Solve $f(x) = 0$.* *(2)*

(d) *Sketch the curve $y = f(x)$, showing the coordinates of any points where it meets the axes and of its turning point.* *(2)*

(e) *Find the set of values of x for which $f(x) < 0$.* *(2)*

Solution.

The idea.

Completing the square shows the turning point immediately; factorising shows the roots; and the sketch then answers the inequality for free.

Step 1. (a) Halve the x coefficient.

For $x^2 + bx$, the completed square is $\left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$. Here $b = -6$, so $\frac{b}{2} = -3$:

$$x^2 - 6x = (x-3)^2 - 9.$$

↪ *Why subtract 9? Because $(x-3)^2 = x^2 - 6x + 9$ has an extra $+9$ that the original didn't have.*

Step 2. (a) Add the constant back.

$$f(x) = (x-3)^2 - 9 + 5 = (x-3)^2 - 4, \qquad p = -3,\ q = -4.$$

Check: $(x-3)^2 - 4 = x^2 - 6x + 9 - 4 = x^2 - 6x + 5$ ✓.

Step 3. (b) Read off the turning point.

$(x-3)^2 \geq 0$, and it equals 0 only when $x = 3$. So the smallest value of $f(x)$ is $0 - 4 = -4$, reached at $x = 3$. The turning point (a minimum) is

$$(3, -4).$$

↪ *Watch the sign: $(x-3)^2$ gives $x = +3$, not -3.*

Step 4. (c) Factorise.

Two numbers that multiply to $+5$ and add to -6: -1 and -5.

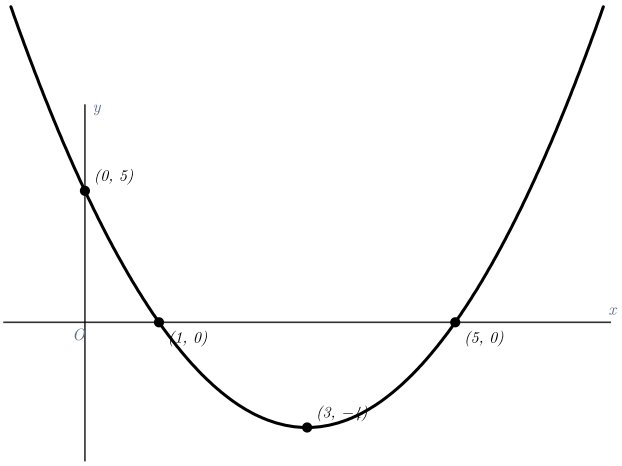
$$x^2 - 6x + 5 = (x-1)(x-5) = 0 \quad \Longrightarrow \quad x = 1 \text{ or } x = 5.$$

Using (a) instead: $(x-3)^2 = 4 \Rightarrow x - 3 = \pm 2 \Rightarrow x = 5$ or $x = 1$. Same answer.

Step 5. (d) Collect the key points.

- The x^2 coefficient is positive, so it is a **U shape** (opens upwards).

- Crosses the x-axis at $(1, 0)$ and $(5, 0)$, from (c).
- Crosses the y-axis where $x = 0$: $f(0) = 5$, so $(0, 5)$.
- Minimum at $(3, -4)$, from (b). It sits exactly halfway between the roots, as it must by symmetry.



Step 6. (e) Use the sketch.

$f(x) < 0$ means "where is the curve **below** the x-axis?". From the sketch, that is the part between the roots:

$$1 < x < 5.$$

↪ *Strict inequality: at $x = 1$ and $x = 5$, $f(x) = 0$, which is not less than 0.*

Answer. (a) $(x - 3)^2 - 4$ (b) $(3, -4)$ (c) $x = 1, 5$ (e) $1 < x < 5$

Common errors.

- Writing the turning point as $(-3, -4)$.
- Answering (e) with "$x < 1$ or $x > 5$" (that is where $f(x) > 0$).

How the marks are earned.

M1 **(a)** $(x - 3)^2$ *($p = -3$)*
A1 $(x - 3)^2 - 4$
B1ft **(b)** $(3, -4)$ *(From their (a).)*
M1 **(c)** Correct method, e.g. $(x - 1)(x - 5) = 0$ or $(x - 3)^2 = 4$
A1 $x = 1,\ x = 5$
B1 **(d)** U-shaped parabola crossing the x-axis at 1 and 5
B1 y-intercept $(0, 5)$ and turning point $(3, -4)$ marked
M1 **(e)** Chooses the region between the roots
A1 $1 < x < 5$ *(Accept $\{x : 1 < x < 5\}$ or $(1, 5)$.)*

∎

Question 6 *Differentiation* 9 marks

The curve C has equation

$$y = 2x^3 - 3x^2 - 12x + 4.$$

(a) *Find* $\dfrac{dy}{dx}$. *(2)*

(b) *Find the coordinates of the stationary points of* C. *(4)*

(c) *Determine the nature of each stationary point, giving a reason for your answer.* *(3)*

Solution.

The idea.

Stationary points are where the gradient is zero, so differentiate, set equal to 0 *and solve. The sign of the second derivative then says whether each one is a minimum (curving up) or a maximum (curving down).*

Step 1. (a) Differentiate term by term.

Multiply by the power, then reduce the power by one. A constant has gradient 0.

$$\frac{d}{dx}(2x^3) = 6x^2, \quad \frac{d}{dx}(-3x^2) = -6x, \quad \frac{d}{dx}(-12x) = -12, \quad \frac{d}{dx}(4) = 0.$$

$$\frac{dy}{dx} = 6x^2 - 6x - 12.$$

Step 2. (b) Set the gradient to zero.

$$6x^2 - 6x - 12 = 0.$$

Divide by the common factor 6:

$$x^2 - x - 2 = 0 \quad \Longrightarrow \quad (x-2)(x+1) = 0 \quad \Longrightarrow \quad x = 2 \text{ or } x = -1.$$

Step 3. (b) Find the y coordinates.

Substitute into the **original** equation (not into $\frac{dy}{dx}$, which is 0 there by design):

$$x = 2: \quad y = 2(8) - 3(4) - 12(2) + 4 = 16 - 12 - 24 + 4 = -16,$$

$$x = -1: \quad y = 2(-1) - 3(1) - 12(-1) + 4 = -2 - 3 + 12 + 4 = 11.$$

The stationary points are $(2, -16)$ and $(-1, 11)$.

↪ *With negative x, put brackets round it:* $(-1)^3 = -1$ *and* $(-1)^2 = +1$.

Step 4. (c) The second derivative.

Differentiate again:

$$\frac{d^2y}{dx^2} = 12x - 6.$$

It measures how the gradient is changing: positive means the curve bends upwards (a valley), negative means it bends downwards (a hill).

Step 5. (c) Test each point.

- At $x = 2$: $\dfrac{d^2y}{dx^2} = 24 - 6 = 18 > 0$, so $(2, -16)$ is a **minimum**.
- At $x = -1$: $\dfrac{d^2y}{dx^2} = -12 - 6 = -18 < 0$, so $(-1, 11)$ is a **maximum**.

This matches the shape of a positive cubic: up to a hill on the left, down into a valley on the right.

> **Answer.** (a) $6x^2 - 6x - 12$ (b) $(2, -16)$, $(-1, 11)$ (c) $(2, -16)$ minimum, $(-1, 11)$ maximum

Another method.
Sign test instead of the second derivative. $\frac{dy}{dx} = 6(x - 2)(x + 1)$. At $x = 0$ it is $-12 < 0$; at $x = 3$ it is $+24 > 0$; at $x = -2$ it is $+24 > 0$. So the gradient goes $+ \to -$ through $x = -1$ (maximum) and $- \to +$ through $x = 2$ (minimum).

Common errors.

- Substituting into $\frac{dy}{dx}$ to find y (always giving 0).
- Stating "minimum" with no reason: the mark needs the sign of $\frac{d^2y}{dx^2}$ (or a sign test).

How the marks are earned.
M1 **(a)** $x^n \to x^{n-1}$ for at least one term
A1 $6x^2 - 6x - 12$
M1 **(b)** Sets their $\dfrac{dy}{dx} = 0$ and solves a three-term quadratic
A1 $x = 2,\ x = -1$
M1 Substitutes an x value into y
A1 $(2, -16)$ and $(-1, 11)$
M1 **(c)** $\dfrac{d^2y}{dx^2} = 12x - 6$ *(Or a correct sign test either side.)*
M1 Evaluates at both points: 18 and -18
A1 $(2, -16)$ minimum, $(-1, 11)$ maximum, with reasons

∎

Question 7 *Integration* 6 marks

$$f'(x) = 3x^2 + 4x - 5$$

(a) *Find* $\displaystyle\int (3x^2 + 4x - 5)\,dx$. *(3)*

(b) *Hence find the area of the region bounded by the curve* $y = 3x^2 + 4x - 5$*, the lines* $x = 1$ *and* $x = 2$*, and the* x*-axis.* *(3)*

Solution.

The idea.

Integration reverses differentiation: raise the power by one and divide by the new power. A definite integral between two limits gives an area (when the curve is above the axis); a known point on the curve fixes the constant.

Step 1. (a) Integrate term by term.

For each x^n: add one to the power, then divide by the new power.

$$\int 3x^2\,dx = \frac{3x^3}{3} = x^3, \qquad \int 4x\,dx = \frac{4x^2}{2} = 2x^2, \qquad \int -5\,dx = -5x.$$

$$\int (3x^2 + 4x - 5)\,dx = x^3 + 2x^2 - 5x + c.$$

↪ *The* $+c$ *matters: every function* $x^3 + 2x^2 - 5x +$ (any constant) *has this derivative.*

Check by differentiating: $3x^2 + 4x - 5$ ✓.

Step 2. (b) Is the region above the axis.

Before using an integral as an area, check the curve doesn't dip below the x-axis between the limits. At $x = 1$: $3 + 4 - 5 = 2 > 0$, and the curve is increasing for $x > 0$ (its gradient $6x + 4$ is positive). So the whole region is above the axis.

Step 3. (b) Evaluate between the limits.

Let $F(x) = x^3 + 2x^2 - 5x$ (no need for c, it cancels).

$$F(2) = 8 + 8 - 10 = 6, \qquad F(1) = 1 + 2 - 5 = -2.$$

$$\text{Area} = \int_1^2 (3x^2 + 4x - 5)\,dx = F(2) - F(1) = 6 - (-2) = 8.$$

↪ *Top limit minus bottom limit, and keep the brackets:* $6 - (-2) = 8$*, not* 4.

Answer. (a) $x^3 + 2x^2 - 5x + c$ (b) 8

Common errors.

- Forgetting $+c$ in (a) (that loses a mark on its own).

- $F(2) - F(1)$ computed as $6 - 2$.

How the marks are earned.

M1 **(a)** $x^n \to x^{n+1}$ for at least one term

A1 Two terms correct (unsimplified)

A1 $x^3 + 2x^2 - 5x + c$ *(Including $+c$.)*

M1 **(b)** Substitutes 2 and 1 into their integral

dM1 Subtracts the right way round: $(8 + 8 - 10) - (1 + 2 - 5)$ *(Dependent on the previous M mark.)*

A1 8

∎

Question 8 *Integration: finding a curve* 3 marks

> *The curve* $y = f(x)$ *has gradient function* $f'(x) = 3x^2 + 4x - 5$ *and passes through the point* $(1, 4)$.
>
> *Find* $f(x)$.

Solution.

The idea.

Integrating the gradient function gives the equation of the curve, but only up to an unknown constant. The point the curve passes through fixes that constant.

Step 1. Use the point to find c.

$f(x)$ is the integral of $f'(x)$, so integrate it term by term (raise each power by one and divide by the new power):

$$f(x) = x^3 + 2x^2 - 5x + c.$$

The curve passes through $(1, 4)$, so $f(1) = 4$:

$$1 + 2 - 5 + c = 4 \quad \Longrightarrow \quad -2 + c = 4 \quad \Longrightarrow \quad c = 6.$$

$$f(x) = x^3 + 2x^2 - 5x + 6.$$

Check: $f(1) = 1 + 2 - 5 + 6 = 4$ ✓.

Answer. $f(x) = x^3 + 2x^2 - 5x + 6$

How the marks are earned.

M1 $f(x) = x^3 + 2x^2 - 5x + c$
M1 Substitutes $(1, 4)$: $1 + 2 - 5 + c = 4$
A1 $f(x) = x^3 + 2x^2 - 5x + 6$

∎

Question 9 *Binomial expansion* 10 marks

(a) *Find the first four terms, in ascending powers of x, of the binomial expansion of $(1+2x)^6$, giving each term in its simplest form.* *(4)*

(b) *Hence find the coefficient of x^2 in the expansion of $(3-x)(1+2x)^6$.* *(3)*

(c) *Use your expansion from (a), with a suitable value of x, to find an estimate for 1.02^6. Give your answer to 5 decimal places.* *(3)*

Solution.

The idea.
$(1+2x)^6 = \sum \binom{6}{r}(2x)^r$. *The key habit is to keep $(2x)$ in a bracket, so the 2 gets raised to the power too.*

Step 1. (a) The general term.
The binomial theorem gives

$$(1+2x)^6 = \binom{6}{0} + \binom{6}{1}(2x) + \binom{6}{2}(2x)^2 + \binom{6}{3}(2x)^3 + \cdots$$

with $\binom{6}{0}=1$, $\binom{6}{1}=6$, $\binom{6}{2}=15$, $\binom{6}{3}=20$ (row 6 of Pascal's triangle: $1, 6, 15, 20, 15, 6, 1$).

Step 2. (a) Simplify each term.

$$\begin{aligned} 6(2x) &= 12x, \\ 15(2x)^2 &= 15 \times 4x^2 = 60x^2, \\ 20(2x)^3 &= 20 \times 8x^3 = 160x^3. \end{aligned}$$

$$(1+2x)^6 = 1 + 12x + 60x^2 + 160x^3 + \cdots$$

↪ *The classic error is $15 \times 2x^2 = 30x^2$: the bracket $(2x)^2$ squares the 2 as well.*

Step 3. (b) Which products give x squared.

$$(3-x)(1+12x+60x^2+\cdots).$$

An x^2 term can only come from

- $3 \times 60x^2 = 180x^2$, or
- $(-x) \times 12x = -12x^2$.

No other pair of terms has powers adding to 2.

Step 4. (b) Add them.

$$180 - 12 = 168.$$

The coefficient of x^2 is 168.

Step 5. (c) Choose x.

I want $1 + 2x = 1.02$, so $2x = 0.02$ and $x = 0.01$.

↪ *A small x is ideal: each later term is much smaller than the one before, so four terms are very accurate.*

Step 6. (c) Substitute.

$$\begin{aligned} 1 + 12(0.01) + 60(0.01)^2 + 160(0.01)^3 &= 1 + 0.12 + 0.006 + 0.00016 \\ &= 1.12616. \end{aligned}$$

A calculator gives $1.02^6 = 1.1261624\ldots$, so the estimate is correct to 5 decimal places.

Answer. (a) $1 + 12x + 60x^2 + 160x^3$ (b) 168 (c) 1.12616

Common errors.

- $15 \times 2x^2$ instead of $15 \times 4x^2$.
- In (b), only multiplying 3×60.
- Choosing $x = 0.02$ in (c).

How the marks are earned.

M1 **(a)** Correct binomial coefficients $\binom{6}{1}, \binom{6}{2}, \binom{6}{3}$ with powers of $2x$ *(Allow $6, 15, 20$.)*
A1 $1 + 12x$
A1 $+60x^2$
A1 $+160x^3$
M1 **(b)** Multiplies the correct pair of terms: $3 \times 60x^2$ and $-x \times 12x$
A1 One of 180 or -12 correct
A1 168
M1 **(c)** $x = 0.01$
M1 Substitutes into their four terms
A1 1.12616 *(cao)*

∎

Question 10 *Trigonometry* 7 marks

(a) *Solve, for* $0 \le x < 360^\circ$, $\sin x = 0.5$. *(2)*

(b) *Solve, for* $0 \le x < 360^\circ$, *(5)*

$$2\cos^2 x + 3\sin x = 3.$$

Solution.

The idea.

For trig equations: get a single trig function (using an identity if needed), solve for that function, then find ***every*** *angle in the range using the graph's symmetry. The triangle uses the cosine rule (two sides and the included angle) and* $\frac{1}{2}ab\sin C$.

Step 1. (a) The principal value.

$\sin^{-1}(0.5) = 30^\circ$. That's the calculator's answer, but there is usually a second one.

Step 2. (a) Use the symmetry of sine.

The graph of $y = \sin x$ is symmetrical about $x = 90^\circ$, so $\sin(180^\circ - x) = \sin x$. The second solution is

$$180^\circ - 30^\circ = 150^\circ.$$

Adding 360° to either takes us out of range, so $x = 30^\circ$ or 150°.

Step 3. (b) Get a single trig function.

The equation mixes $\cos^2 x$ and $\sin x$. Use $\sin^2 x + \cos^2 x = 1$ to replace $\cos^2 x$ by $1 - \sin^2 x$:

$$2(1 - \sin^2 x) + 3\sin x = 3.$$

Step 4. (b) Rearrange into a quadratic.

$$2 - 2\sin^2 x + 3\sin x - 3 = 0 \quad \Longrightarrow \quad -2\sin^2 x + 3\sin x - 1 = 0.$$

Multiply by -1 so the leading term is positive:

$$2\sin^2 x - 3\sin x + 1 = 0.$$

This is a quadratic in $s = \sin x$: $2s^2 - 3s + 1 = 0$.

Step 5. (b) Solve the quadratic.

$$2s^2 - 3s + 1 = (2s - 1)(s - 1) = 0 \quad \Longrightarrow \quad s = \tfrac{1}{2} \quad \text{or} \quad s = 1.$$

Step 6. (b) Find all the angles.

- $\sin x = \frac{1}{2}$: $x = 30^\circ$ or 150°, exactly as in (a).
- $\sin x = 1$: $x = 90^\circ$ only (the top of the sine wave, where there's only one solution per turn).

So $x = 30^\circ,\ 90^\circ,\ 150^\circ$. **Check** $x = 90^\circ$**:** $2\cos^2 90^\circ + 3\sin 90^\circ = 0 + 3 = 3$ ✓.

Answer. (a) $30^\circ, 150^\circ$ (b) $30^\circ, 90^\circ, 150^\circ$

Common errors.

- Missing 150° (only giving the calculator's answer).
- Replacing $\cos^2 x$ with $1 + \sin^2 x$.
- Dividing by $\sin x$, which loses solutions; always factorise instead.

How the marks are earned.

B1 (a) 30°
B1 150° *(And no extra answers in range.)*
M1 (b) Uses $\cos^2 x = 1 - \sin^2 x$
A1 $2\sin^2 x - 3\sin x + 1 = 0$ *(Or any equivalent three-term quadratic.)*
M1 Solves the quadratic in $\sin x$
A1 $\sin x = 1,\ \sin x = \frac{1}{2}$
A1 $x = 30^\circ,\ 90^\circ,\ 150^\circ$ *(All three, no extras.)*

∎

Question 11 *Triangles* 4 marks

In triangle ABC, $AB = 7$ cm, $AC = 5$ cm and angle $BAC = 60^\circ$.

(a) *Find the length of BC, giving your answer to 3 significant figures.* *(2)*

(b) *Find the exact area of triangle ABC.* *(2)*

Solution.

The idea.

*Two sides and the angle **between** them: the cosine rule gives the third side, and $\frac{1}{2}ab\sin C$ gives the area.*

Step 1. (a) Cosine rule.

I know two sides and the angle **between** them, so use the cosine rule for the side opposite that angle:

$$BC^2 = AB^2 + AC^2 - 2(AB)(AC)\cos A = 49 + 25 - 2(7)(5)\left(\tfrac{1}{2}\right) = 74 - 35 = 39.$$

$$BC = \sqrt{39} = 6.2449\ldots \approx 6.24 \text{ cm}.$$

Step 2. (b) Area formula.

$$\text{Area} = \frac{1}{2}ab\sin C = \frac{1}{2} \times 7 \times 5 \times \sin 60^\circ = \frac{35}{2} \times \frac{\sqrt{3}}{2} = \frac{35\sqrt{3}}{4} \text{ cm}^2.$$

↪ *"Exact" means keep the surd: $\sin 60^\circ = \frac{\sqrt{3}}{2}$. As a decimal it would be 15.2 cm², but that isn't exact.*

Answer. (a) 6.24 cm (b) $\dfrac{35\sqrt{3}}{4}$ cm²

How the marks are earned.

M1 (a) $BC^2 = 7^2 + 5^2 - 2(7)(5)\cos 60^\circ$
A1 $BC = 6.24$ cm *(From $\sqrt{39}$.)*
M1 (b) $\frac{1}{2} \times 7 \times 5 \times \sin 60^\circ$
A1 $\dfrac{35\sqrt{3}}{4}$ cm²

∎

Question 12 *Exponentials and logarithms* 6 marks

> ***(a)*** *Solve* $3^x = 20$*, giving your answer to 3 significant figures.* *(2)*
>
> ***(b)*** *Solve* $\log_2 x + \log_2(x+2) = 3$*.* *(4)*

Solution.

The idea.

Logarithms undo powers. When the unknown is in the power, take logs (or $\ln$*) of both sides. When the unknown is inside logs, combine them into one log and then rewrite as a power.*

Step 1. (a) Take logs of both sides.

$$\ln(3^x) = \ln 20 \quad\Longrightarrow\quad x\ln 3 = \ln 20 \quad\Longrightarrow\quad x = \frac{\ln 20}{\ln 3} = \frac{2.9957\ldots}{1.0986\ldots} = 2.7268\ldots \approx 2.73.$$

↪ *Sense check:* $3^2 = 9$ *and* $3^3 = 27$*, so* x *should be between 2 and 3, closer to 3.* ✓

Step 2. (b) Combine the logs.

The law $\log_a m + \log_a n = \log_a(mn)$ gives

$$\log_2 x(x+2) = 3.$$

Step 3. (b) Rewrite as a power.

$\log_2 N = 3$ means $N = 2^3$:

$$x(x+2) = 8 \quad\Longrightarrow\quad x^2 + 2x - 8 = 0 \quad\Longrightarrow\quad (x+4)(x-2) = 0.$$

So $x = 2$ or $x = -4$.

Step 4. (b) Reject the impossible root.

The original equation contains $\log_2 x$, which only exists for $x > 0$. So $x = -4$ is **not** a solution.

$$x = 2.$$

Check: $\log_2 2 + \log_2 4 = 1 + 2 = 3$ ✓.

Answer. (a) 2.73 (b) $x = 2$

Common errors.

- Writing $\log_2 x + \log_2(x+2) = \log_2(2x+2)$.
- Keeping $x = -4$.

How the marks are earned.

M1 (a) $x = \dfrac{\ln 20}{\ln 3}$ or $\log_3 20$

A1 2.73

M1 (b) Uses $\log a + \log b = \log ab$: $\log_2 x(x+2) = 3$

M1 Removes the log: $x(x+2) = 2^3$

A1 $x^2 + 2x - 8 = 0$

A1 $x = 2$ only, with $x = -4$ rejected

∎

Question 13 *Exponential models* 6 marks

> *The number of bacteria in a culture is modelled by $P = 200e^{0.03t}$, where t is the time in hours.*
>
> *(i) State the initial number of bacteria. (ii) Find the time taken for the number of bacteria to reach 500, giving your answer to 3 significant figures. (iii) Find the rate at which the number of bacteria is increasing when $t = 10$.*

Solution.

The idea.

Read the model carefully. "Initially" means $t = 0$; "when does it reach 500" means solve an exponential equation with $\ln$; and "rate of increase" means the derivative $\frac{dP}{dt}$.

Step 1. (i) Initial number.

"Initially" means $t = 0$: $P = 200e^{0} = 200$.

Step 2. (ii) Set P to 500.

$$200e^{0.03t} = 500 \quad \Longrightarrow \quad e^{0.03t} = 2.5.$$

Take $\ln$ of both sides (it undoes e):

$$0.03t = \ln 2.5 \quad \Longrightarrow \quad t = \frac{\ln 2.5}{0.03} = \frac{0.91629\ldots}{0.03} = 30.54\ldots \approx 30.5 \text{ hours}.$$

Step 3. (iii) Differentiate.

The rate of increase is $\frac{dP}{dt}$. Using $\frac{d}{dt}e^{kt} = ke^{kt}$:

$$\frac{dP}{dt} = 200 \times 0.03\, e^{0.03t} = 6e^{0.03t}.$$

At $t = 10$:

$$\frac{dP}{dt} = 6e^{0.3} = 6 \times 1.34986\ldots = 8.099\ldots \approx 8.10 \text{ bacteria per hour}.$$

↪ *Notice $\frac{dP}{dt} = 0.03P$: the growth rate is always 3% of the current population. That is what exponential growth means.*

Answer. (i) 200 (ii) 30.5 hours (iii) 8.10 per hour

Common errors.

- In (iii), giving the value of P at $t = 10$ instead of $\frac{dP}{dt}$.
- In (ii), dividing by 200 incorrectly: $\frac{500}{200} = 2.5$.

How the marks are earned.

B1 (i) 200

M1 (ii) $200e^{0.03t} = 500$

M1 $0.03t = \ln 2.5$

A1 $t = 30.5$ hours

M1 (iii) $\dfrac{dP}{dt} = 6e^{0.03t}$

A1 8.10 bacteria per hour

∎

Question 14 *Circles* 8 marks

The circle C has equation

$$x^2 + y^2 - 6x + 4y - 12 = 0.$$

(a) *Find the coordinates of the centre of C and the radius of C.* *(3)*

(b) *Verify that the point $P(6, 2)$ lies on C.* *(1)*

(c) *Find an equation of the tangent to C at P, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers.* *(4)*

Solution.

The idea.

*Completing the square turns the circle into centre–radius form. A tangent is perpendicular to the radius at the point of contact. And a line is tangent to a circle exactly when substituting gives a quadratic with a **repeated** root: discriminant zero.*

Step 1. (a) Group the x and y terms.

$$(x^2 - 6x) + (y^2 + 4y) = 12.$$

Step 2. (a) Complete both squares.

$$x^2 - 6x = (x - 3)^2 - 9, \qquad y^2 + 4y = (y + 2)^2 - 4.$$

So

$$(x - 3)^2 - 9 + (y + 2)^2 - 4 = 12 \quad \Longrightarrow \quad (x - 3)^2 + (y + 2)^2 = 25.$$

Step 3. (a) Read off centre and radius.

Compare with $(x - a)^2 + (y - b)^2 = r^2$:

$$\text{centre } (3, -2), \qquad r = \sqrt{25} = 5.$$

↪ *Signs flip:* $(y + 2)$ *means* $b = -2$.

Step 4. (b) Substitute P.

$$6^2 + 2^2 - 6(6) + 4(2) - 12 = 36 + 4 - 36 + 8 - 12 = 0. \checkmark$$

So P lies on C. (Equivalently, P is $(3, 4)$ from the centre, and $\sqrt{9 + 16} = 5$.)

Step 5. (c) Gradient of the radius CP.

$$m_{CP} = \frac{2 - (-2)}{6 - 3} = \frac{4}{3}.$$

Step 6. (c) The tangent is perpendicular.

A tangent meets the radius at a right angle, so its gradient is the negative reciprocal:

$$m_{\text{tangent}} = -\frac{3}{4}.$$

Step 7. (c) Equation of the tangent.

Through $P(6, 2)$:

$$y - 2 = -\frac{3}{4}(x - 6).$$

Multiply by 4: $4y - 8 = -3x + 18$, so

$$3x + 4y - 26 = 0.$$

Check P: $18 + 8 - 26 = 0$ ✓.

> **Answer.** (a) centre $(3, -2)$, radius 5 (c) $3x + 4y - 26 = 0$

Common errors.

- Centre $(-3, 2)$: the signs flip when reading off.
- Radius 25 (forgetting the square root) or $\sqrt{12}$.
- Using the radius gradient, not the perpendicular one, for the tangent.

How the marks are earned.

M1 **(a)** Attempts $(x - 3)^2 + (y + 2)^2 = \ldots$
A1 Centre $(3, -2)$
A1 Radius 5
B1 **(b)** $36 + 4 - 36 + 8 - 12 = 0$ shown *(Or $3^2 + 4^2 = 25$.)*
M1 **(c)** Gradient of radius $= \dfrac{2 - (-2)}{6 - 3} = \dfrac{4}{3}$
M1 Tangent gradient $-\dfrac{3}{4}$ *(Negative reciprocal.)*
M1 $y - 2 = -\dfrac{3}{4}(x - 6)$
A1 $3x + 4y - 26 = 0$

∎

Question 15 *Circles and tangent lines* 9 marks

The circle C has equation $(x-3)^2+(y+2)^2=25$. The line with equation $y=x+k$, where k is a constant, is a tangent to C.

(a) *Find the two possible exact values of k.* *(7)*

(b) *Hence find the set of values of k for which the line $y=x+k$ meets C at two distinct points.* *(2)*

A video explanation of this question is at lemmamathematics.com/papers/pure-1#q15.

Solution.

The idea.

*A line meets a circle in 0, 1 or 2 points. Substituting the line into the circle gives a quadratic, and the line is a **tangent** exactly when that quadratic has one repeated root, i.e. its discriminant is zero. The video shows the line sliding across the circle.*

Step 1. (a) The tangency condition.

A line and a circle can meet at 0, 1 or 2 points. Substituting the line into the circle gives a quadratic, and **one** meeting point (a tangent) means that quadratic has **one repeated root**, so its discriminant is 0. Watch the video above to see the line slide across the circle.

Step 2. (a) Substitute the line.

Put $y=x+k$ into $x^2+y^2-6x+4y-12=0$:

$$x^2+(x+k)^2-6x+4(x+k)-12=0.$$

Expand: $(x+k)^2=x^2+2kx+k^2$, so

$$x^2+x^2+2kx+k^2-6x+4x+4k-12=0.$$

Collect powers of x:

$$2x^2+(2k-2)x+(k^2+4k-12)=0.$$

So $a=2$, $b=2k-2$, $c=k^2+4k-12$.

Step 3. (a) Set the discriminant to zero.

$$b^2-4ac=(2k-2)^2-4(2)(k^2+4k-12)=0.$$

Expand each piece:

$$(2k-2)^2=4k^2-8k+4, \qquad 8(k^2+4k-12)=8k^2+32k-96.$$

$$4k^2-8k+4-8k^2-32k+96=0 \quad\Longrightarrow\quad -4k^2-40k+100=0.$$

Divide by -4:

$$k^2 + 10k - 25 = 0.$$

Step 4. (a) Solve for k.

It doesn't factorise, so complete the square (or use the formula):

$$(k+5)^2 - 25 - 25 = 0 \quad \Longrightarrow \quad (k+5)^2 = 50 \quad \Longrightarrow \quad k + 5 = \pm\sqrt{50} = \pm 5\sqrt{2}.$$

$$k = -5 + 5\sqrt{2} \quad \text{or} \quad k = -5 - 5\sqrt{2}.$$

(Numerically $k \approx 2.07$ or $k \approx -12.07$.)

Step 5. (a) Check with distances.

A line is tangent exactly when its distance from the centre equals the radius. The line $x - y + k = 0$ is at distance

$$\frac{|3 - (-2) + k|}{\sqrt{1^2 + 1^2}} = \frac{|5 + k|}{\sqrt{2}}$$

from $(3, -2)$. Setting this equal to 5: $|5 + k| = 5\sqrt{2}$, so $k = -5 \pm 5\sqrt{2}$. ✓ Two completely different methods agree.

Step 6. (b) Two points means inside.

The line meets C twice when the discriminant is **positive**: $-4k^2 - 40k + 100 > 0$, i.e. $k^2 + 10k - 25 < 0$. This upside-down condition holds **between** the roots:

$$-5 - 5\sqrt{2} < k < -5 + 5\sqrt{2}.$$

Geometrically: slide the line $y = x + k$ up and down; it cuts the circle twice exactly while it is between the two tangent positions.

Answer. (a) $k = -5 \pm 5\sqrt{2}$ (b) $-5 - 5\sqrt{2} < k < -5 + 5\sqrt{2}$

Common errors.

- Errors expanding $(2k - 2)^2$; write it out as $(2k - 2)(2k - 2)$ if unsure.
- Dropping a term when substituting: $4y$ becomes $4(x + k) = 4x + 4k$, not $4x + k$.
- Answering (b) with the outside region, which is where the line misses the circle.

Examiner's note.

Two marks in (b) for one line of work: once you have the tangent values, "two intersections" is always the region between them.

How the marks are earned.

M1 **(a)** Substitutes $y = x + k$ into the equation of C

A1 $2x^2 + (2k - 2)x + k^2 + 4k - 12 = 0$

M1 Uses $b^2 - 4ac = 0$

A1 $k^2 + 10k - 25 = 0$ or equivalent

M1 Solves their quadratic in k

A1 $k = -5 + 5\sqrt{2}$

A1 $k = -5 - 5\sqrt{2}$

M1 **(b)** Chooses the inside region

A1 $-5 - 5\sqrt{2} < k < -5 + 5\sqrt{2}$

∎