Pure Mathematics

Paper 2 · Standard: model solutions

LP2/P · Level 2

How to use these solutions. Try each question under timed conditions first, then mark it with the mark scheme. Read the full solution for every question, even the ones you got right: they often show a faster method, a check, or a common error to avoid. Each solution begins with *the idea*, then works through numbered steps; remarks marked ↪ explain *why*. Each ends with the boxed answer and how the marks are earned.

Question 1 *Inequalities* 6 marks

Find the set of values of x *for which*

(a) $x^2 - 2x - 15 > 0$, *(3)*

(b) $3x - 1 < 2x + 7$, *(1)*

(c) *both* $x^2 - 2x - 15 > 0$ *and* $3x - 1 < 2x + 7$, *giving your answer in set notation.* *(2)*

Solution.

The idea.

Solve each inequality on its own, then find where ***both*** *are true at once. A number line is the clearest way to combine them.*

Step 1. (a) Find the critical values.

Solve the related equation $x^2 - 2x - 15 = 0$. Two numbers that multiply to -15 and add to -2: -5 and $+3$.

$$(x - 5)(x + 3) = 0 \quad \Longrightarrow \quad x = 5 \text{ or } x = -3.$$

Step 2. (a) Choose the region.

$y = x^2 - 2x - 15$ is a U-shaped parabola crossing the axis at -3 and 5. It is **above** the axis (> 0) outside the roots:

$$x < -3 \quad \text{or} \quad x > 5.$$

Check a value: $x = 0$ gives -15, not > 0, so the middle is correctly excluded.

Step 3. (b) A linear inequality.

Subtract $2x$ from both sides and add 1:

$$3x - 2x < 7 + 1 \quad \Longrightarrow \quad x < 8.$$

Step 4. (c) Combine on a number line.

I need x in (a)'s region **and** in (b)'s region.

- The part $x < -3$ is entirely inside $x < 8$: keep all of it.
- The part $x > 5$ overlaps $x < 8$ only for $5 < x < 8$.

$$\{x : x < -3\} \cup \{x : 5 < x < 8\}.$$

↪ *"Set notation" means braces and $\cup$ (union). Writing "or" in words may not get the final mark.*

Answer. (a) $x < -3$ or $x > 5$ (b) $x < 8$ (c) $\{x : x < -3\} \cup \{x : 5 < x < 8\}$

Common errors.

- Answering (a) with $-3 < x < 5$ (that's where the quadratic is negative).
- In (c), writing $x < 8$ and $x > 5$ and forgetting the $x < -3$ part.

How the marks are earned.

M1 **(a)** Critical values from $(x - 5)(x + 3) = 0$
A1 $x = 5,\ x = -3$
A1ft $x < -3$ or $x > 5$ *(Outside region of their critical values.)*
B1 **(b)** $x < 8$
M1 **(c)** Combines their (a) and (b) correctly
A1 $\{x : x < -3\} \cup \{x : 5 < x < 8\}$

∎

Question 2 *Proof* 4 marks

> *Prove that, for all integers n, $n^3 - n$ is divisible by 6.*

Solution.

The idea.

*A proof must work for **every** integer, so I can't just test examples. Factorising reveals a structure (three consecutive integers) that forces the divisibility. A false "for all" statement needs only **one** counterexample.*

Step 1. Factorise.

Take out the common factor n, then use the difference of two squares:

$$n^3 - n = n(n^2 - 1) = n(n-1)(n+1) = (n-1)\,n\,(n+1).$$

Step 2. Recognise the structure.

$(n-1)$, n, $(n+1)$ are **three consecutive integers**, for example $4, 5, 6$ when $n = 5$.

Step 3. Why there is a factor 2.

In any two consecutive integers, one is even. So at least one of the three is even, and the product is a multiple of 2.

Step 4. Why there is a factor 3.

Every integer leaves remainder 0, 1 or 2 when divided by 3, and three consecutive integers have three **different** remainders. So exactly one of them is a multiple of 3, and the product is a multiple of 3.

Step 5. Conclude.

The product is a multiple of 2 and of 3. Because 2 and 3 are coprime (no common factor), it is a multiple of $2 \times 3 = 6$. Therefore $n^3 - n$ is divisible by 6 for every integer n. ■

↪ *The "coprime" step matters: being divisible by 2 and by 4 does not make a number divisible by 8 (think of 4).*

Answer. Proof as above.

Another method.

Proof by cases. Every integer is $6k + r$ with $r \in \{0, 1, 2, 3, 4, 5\}$. You could check all six cases, but the consecutive-integers argument is shorter and explains *why* it's true.

Common errors.

- "It works for $n = 1, 2, 3, 4$, so it's true": checking examples is not a proof.
- Saying "one of them is divisible by 6".

How the marks are earned.

M1 $n^3 - n = n(n^2 - 1) = (n - 1)n(n + 1)$

A1 States these are three consecutive integers

M1 Argues one of them is even and one is a multiple of 3

A1 Concludes: divisible by $\mathbf{2 \times 3 = 6}$, because 2 and 3 have no common factor *(Complete, convincing proof.)*

∎

Question 3 *Disproof by counterexample* 2 marks

> *Show, by means of a counterexample, that the statement "$2^n + 1$ is prime for all positive integers n" is not true.*

Solution.

The idea.

*A statement claimed for **all** positive integers is disproved by a single value where it fails. Try small values of n until one breaks, and say clearly why it breaks.*

Step 1. Try small values.

$n = 1$: 3, prime. $n = 2$: 5, prime. $n = 3$: $2^3 + 1 = 9$.

Step 2. State the counterexample.

$9 = 3 \times 3$ is **not** prime. So $n = 3$ is a counterexample and the statement is false.

↪ *Write down why it fails. "$n = 3$ gives 9" alone isn't enough; say that 9 has a factor of 3.*

Answer. $n = 3$: $2^3 + 1 = 9 = 3 \times 3$, which is not prime.

How the marks are earned.

M1 Tests a value of n and evaluates $2^n + 1$

A1 $n = 3$: $2^3 + 1 = 9 = 3 \times 3$, not prime *(Must say why it isn't prime.)*

∎

Question 4 *Functions* 8 marks

The functions f and g are defined by

$$f(x) = 2x + 3, \quad x \in \mathbb{R}, \qquad g(x) = x^2 - 1, \quad x \in \mathbb{R}.$$

(a) *Find an expression for $fg(x)$.* *(2)*

(b) *Solve the equation $fg(x) = gf(x)$, giving your answers in simplest surd form.* *(4)*

(c) *Find $f^{-1}(x)$.* *(2)*

Solution.

The idea.

$fg(x)$ means "do g first, then f" (the function nearest x acts first). Composition is not commutative, so fg and gf are usually different, and the question asks where they happen to agree.

Step 1. (a) Apply g then f.

$g(x) = x^2 - 1$. Now put that whole expression into f, wherever f has an x:

$$fg(x) = f(x^2 - 1) = 2(x^2 - 1) + 3 = 2x^2 - 2 + 3 = 2x^2 + 1.$$

Step 2. (b) Apply f then g.

$$gf(x) = g(2x + 3) = (2x + 3)^2 - 1 = 4x^2 + 12x + 9 - 1 = 4x^2 + 12x + 8.$$

Step 3. (b) Set them equal.

$$2x^2 + 1 = 4x^2 + 12x + 8 \quad \Longrightarrow \quad 0 = 2x^2 + 12x + 7.$$

Step 4. (b) Quadratic formula.

With $a = 2,\ b = 12,\ c = 7$:

$$x = \frac{-12 \pm \sqrt{144 - 56}}{4} = \frac{-12 \pm \sqrt{88}}{4}.$$

Simplify the surd: $\sqrt{88} = \sqrt{4 \times 22} = 2\sqrt{22}$, so

$$x = \frac{-12 \pm 2\sqrt{22}}{4} = \frac{-6 \pm \sqrt{22}}{2}.$$

(Numerically $x \approx -0.655$ or $x \approx -5.345$.)

Step 5. (b) Check one root.

$x = \frac{-6+\sqrt{22}}{2} \approx -0.6548$: $fg = 2(0.4287) + 1 = 1.857$ and $gf = (1.6904)^2 - 1 = 1.857$ ✓.

Step 6. (c) Swap and rearrange.

Write $y = 2x + 3$ and make x the subject:

$$y - 3 = 2x \quad \Longrightarrow \quad x = \frac{y - 3}{2}.$$

So $f^{-1}(x) = \dfrac{x-3}{2}$. **Check:** $f^{-1}(f(x)) = \frac{(2x+3)-3}{2} = x$ ✓. The inverse undoes "double then add 3" by "subtract 3 then halve", in reverse order.

Answer. (a) $2x^2+1$ (b) $x = \dfrac{-6 \pm \sqrt{22}}{2}$ (c) $f^{-1}(x) = \dfrac{x-3}{2}$

Common errors.

- Doing f first for $fg(x)$.
- $(2x+3)^2 = 4x^2 + 9$.
- Leaving $\sqrt{88}$ unsimplified when "simplest surd form" is asked for.

How the marks are earned.

M1 (a) $2(x^2-1)+3$
A1 $2x^2+1$
M1 (b) $gf(x) = (2x+3)^2 - 1$
A1 $4x^2+12x+8$
M1 Sets equal and solves $2x^2+12x+7=0$
A1 $x = \dfrac{-6 \pm \sqrt{22}}{2}$
M1 (c) $y = 2x+3 \Rightarrow x = \dfrac{y-3}{2}$
A1 $f^{-1}(x) = \dfrac{x-3}{2}$

∎

Question 5 *Arithmetic series* 8 marks

An arithmetic series has first term a and common difference d. The 5th term is 17 and the sum of the first 10 terms is 185.

(a) *Find the values of a and d.* *(4)*

(b) *Find the least value of n for which the sum of the first n terms exceeds 1000.* *(4)*

Solution.

The idea.

*Turn each piece of information into an equation using $u_n = a + (n-1)d$ and $S_n = \frac{n}{2}\left(2a + (n-1)d\right)$, then solve. For "least n", solve the inequality and round **up**.*

Step 1. (a) The 5th term.

$u_5 = a + 4d$ (four steps of d after the first term):

$$a + 4d = 17. \qquad (1)$$

Step 2. (a) The sum of 10 terms.

$$S_{10} = \frac{10}{2}(2a + 9d) = 5(2a + 9d) = 185 \quad \Longrightarrow \quad 2a + 9d = 37. \qquad (2)$$

Step 3. (a) Solve simultaneously.

Double (1): $2a + 8d = 34$. Subtract from (2):

$$(2a + 9d) - (2a + 8d) = 37 - 34 \quad \Longrightarrow \quad d = 3.$$

Then $a = 17 - 4(3) = 5$. **Check:** the series is $5, 8, 11, 14, 17, \ldots$, so $u_5 = 17$ ✓, and $S_{10} = 5(10 + 27) = 185$ ✓.

Step 4. (b) Write S n.

$$S_n = \frac{n}{2}\left(10 + 3(n-1)\right) = \frac{n}{2}(3n + 7) = \frac{3n^2 + 7n}{2}.$$

Step 5. (b) Form the inequality.

$$\frac{3n^2 + 7n}{2} > 1000 \quad \Longrightarrow \quad 3n^2 + 7n - 2000 > 0.$$

Step 6. (b) Solve the equation.

$$n = \frac{-7 \pm \sqrt{49 + 24000}}{6} = \frac{-7 \pm \sqrt{24049}}{6} = \frac{-7 \pm 155.08\ldots}{6}.$$

The positive root is $n = 24.68\ldots$. The quadratic is positive beyond it, so I need $n > 24.68\ldots$.

Step 7. (b) Round up and check.

n must be a whole number, so the least is $n = 25$. Check either side:

$$S_{24} = \frac{24 \times 79}{2} = 948 < 1000, \qquad S_{25} = \frac{25 \times 82}{2} = 1025 > 1000. \checkmark$$

↪ *Checking the two integers either side guards against rounding the wrong way.*

Answer. (a) $a = 5$, $d = 3$ (b) $n = 25$

Common errors.

- $u_5 = a + 5d$.
- Rounding 24.68 down to 24.

How the marks are earned.

M1 (a) $a + 4d = 17$
M1 $\frac{10}{2}(2a + 9d) = 185$
M1 Solves simultaneously
A1 $a = 5,\ d = 3$
M1 (b) $\frac{n}{2}\left(10 + 3(n-1)\right) > 1000$
M1 $3n^2 + 7n - 2000 > 0$ *(Three-term quadratic.)*
M1 Solves: $n > 24.68\ldots$
A1 $n = 25$ *(Accept with $S_{24} = 948$, $S_{25} = 1025$ shown.)*

∎

Question 6 *Geometric series* 8 marks

A geometric series has first term 80 and common ratio r, where $0 < r < 1$. The sum to infinity of the series is 320.

(a) *Show that $r = \frac{3}{4}$.* *(2)*

(b) *Find the 4th term of the series.* *(2)*

(c) *Find the smallest value of n for which the sum to infinity exceeds the sum of the first n terms by less than 1.* *(4)*

Solution.

The idea.

Use $S_\infty = \dfrac{a}{1-r}$ and $S_n = \dfrac{a(1-r^n)}{1-r}$. The difference $S_\infty - S_n$ simplifies beautifully, and logs bring n down from the power.

Step 1. (a) Sum to infinity.

$$\frac{80}{1-r} = 320 \quad \Longrightarrow \quad 1 - r = \frac{80}{320} = \frac{1}{4} \quad \Longrightarrow \quad r = \frac{3}{4}. \quad \blacksquare$$

It's valid because $|r| < 1$.

Step 2. (b) The 4th term.

$u_n = ar^{n-1}$, so

$$u_4 = 80\left(\tfrac{3}{4}\right)^3 = 80 \times \frac{27}{64} = \frac{2160}{64} = 33.75.$$

Step 3. (c) Simplify the difference.

$$S_\infty - S_n = \frac{a}{1-r} - \frac{a(1-r^n)}{1-r} = \frac{a\,r^n}{1-r} = S_\infty\, r^n = 320\left(\tfrac{3}{4}\right)^n.$$

↪ *This is worth remembering: what's still "missing" after n terms is just S_∞ shrunk by r^n.*

Step 4. (c) Set up the inequality.

$$320\left(\tfrac{3}{4}\right)^n < 1 \quad \Longrightarrow \quad \left(\tfrac{3}{4}\right)^n < \frac{1}{320}.$$

Step 5. (c) Take logs carefully.

Take $\ln$ of both sides:

$$n \ln \tfrac{3}{4} < \ln \tfrac{1}{320} = -\ln 320.$$

Now $\ln \frac{3}{4}$ is **negative**, so dividing by it **reverses** the inequality:

$$n > \frac{-\ln 320}{\ln 0.75} = \frac{\ln 320}{\ln(4/3)} = \frac{5.768\ldots}{0.2876\ldots} = 20.05\ldots$$

Step 6. (c) The smallest integer.

So $n = 21$. Check: $320(0.75)^{20} = 1.015\ldots > 1$, and $320(0.75)^{21} = 0.761\ldots < 1$ ✓.

Answer. (b) 33.75 (c) $n = 21$

Common errors.

- Forgetting to reverse the inequality when dividing by $\ln 0.75$; this gives the nonsense "$n < 20.05$".
- $u_4 = 80 \times 0.75^4$.

How the marks are earned.

M1 (a) $\dfrac{80}{1-r} = 320$
A1* $1 - r = \frac{1}{4} \Rightarrow r = \frac{3}{4}$ *(Given answer: must be shown.)*
M1 (b) $80 \times \left(\frac{3}{4}\right)^3$
A1 33.75
M1 (c) $S_\infty - S_n = 320\left(\frac{3}{4}\right)^n$
M1 $320\left(\frac{3}{4}\right)^n < 1$
M1 Uses logs: $n > \dfrac{\ln 320}{\ln \frac{4}{3}}$ *(Take care:* $\ln \frac{3}{4} < 0$ *reverses the inequality.)*
A1 $n = 21$

∎

Question 7 *Exponential models* 10 marks

The value, £V, of a car t years after it was first sold is modelled by

$$V = 18\,000e^{-0.15t} + 2000, \qquad t \geq 0.$$

(a) *State the value of the car when it was first sold.* *(1)*

(b) *Find the value of the car after 5 years, to the nearest pound.* *(2)*

(c) *Find the value of t when the car's value first falls below £6000. Give your answer to 2 decimal places.* *(3)*

(d) *Find the rate at which the value of the car is decreasing when $t = 5$.* *(3)*

(e) *State the long-term value of the car according to this model.* *(1)*

Solution.

The idea.

An exponential decay plus a constant: the $e^{-0.15t}$ part shrinks towards zero, so V falls towards £2000. Each part is "substitute", "solve with ln" or "differentiate".

Step 1. (a) Put t equal to 0.

$$V = 18\,000e^{0} + 2000 = 18\,000 + 2000 = £20\,000.$$

Step 2. (b) Put t equal to 5.

$$V = 18\,000e^{-0.75} + 2000 = 18\,000 \times 0.47236\ldots + 2000 = 8502.6\ldots + 2000 = £10\,503.$$

Step 3. (c) Isolate the exponential.

$$18\,000e^{-0.15t} + 2000 = 6000 \quad \Longrightarrow \quad e^{-0.15t} = \frac{4000}{18\,000} = \frac{2}{9}.$$

Step 4. (c) Take ln.

$$-0.15t = \ln\tfrac{2}{9} \quad \Longrightarrow \quad t = \frac{\ln(9/2)}{0.15} = \frac{1.50407\ldots}{0.15} = 10.027\ldots \approx 10.03.$$

Because V is decreasing, it is above £6000 before this time and below it after.

Step 5. (d) Differentiate.

$\frac{d}{dt}e^{-0.15t} = -0.15e^{-0.15t}$, and the constant 2000 disappears:

$$\frac{dV}{dt} = 18\,000 \times (-0.15)e^{-0.15t} = -2700e^{-0.15t}.$$

Step 6. (d) Evaluate at t equal to 5.

$$\frac{dV}{dt} = -2700e^{-0.75} = -1275.4\ldots$$

The negative sign means the value is going **down**: the car is losing value at about **£1275 per year** at that moment.

Step 7. (e) Long-term value.

As $t \to \infty$, $e^{-0.15t} \to 0$, so $V \to 2000$. The model predicts the car never drops below £2000 (perhaps its scrap value).

> **Answer.** (a) £20 000 (b) £10 503 (c) $t = 10.03$ (d) £1275 per year (e) £2000

Common errors.

- Answering (a) with £18 000 (forgetting the $+2000$).
- In (c), taking $\ln$ before subtracting 2000: $\ln(a+b) \neq \ln a + \ln b$.

How the marks are earned.

B1 **(a)** £20 000
M1 **(b)** $18\,000e^{-0.75} + 2000$
A1 £10 503
M1 **(c)** $18\,000e^{-0.15t} = 4000$
M1 $-0.15t = \ln\frac{2}{9}$
A1 $t = 10.03$
M1 **(d)** Differentiates: $ke^{-0.15t}$
A1 $\dfrac{dV}{dt} = -2700e^{-0.15t}$
A1 £1275 per year *(Decreasing at about £1275 per year.)*
B1 **(e)** £2000

∎

Question 8 *Trigonometric identities* 8 marks

(a) *Show that* *(3)*

$$\frac{1-\cos 2\theta}{\sin 2\theta} \equiv \tan\theta.$$

(b) *Solve, for* $0 \le \theta \le \pi$, *(5)*

$$3\sin 2\theta = 2\cos\theta,$$

giving your answers to 3 decimal places where appropriate.

Solution.

The idea.

Double-angle formulae convert 2θ *into* θ. *In (b), the big temptation is to divide by* $\cos\theta$*; factorising instead keeps the solution* $\cos\theta = 0$.

Step 1. (a) Choose the right cos 2θ.

There are three versions of $\cos 2\theta$. I want the 1 to cancel, so I use $\cos 2\theta = 1 - 2\sin^2\theta$:

$$1 - \cos 2\theta = 1 - (1 - 2\sin^2\theta) = 2\sin^2\theta.$$

Step 2. (a) Use sin 2θ.

$\sin 2\theta = 2\sin\theta\cos\theta$, so

$$\frac{1-\cos 2\theta}{\sin 2\theta} = \frac{2\sin^2\theta}{2\sin\theta\cos\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta. \quad \blacksquare$$

Step 3. (b) Expand sin 2θ.

$$3(2\sin\theta\cos\theta) = 2\cos\theta \quad \Longrightarrow \quad 6\sin\theta\cos\theta - 2\cos\theta = 0.$$

Step 4. (b) Factorise.

$$2\cos\theta\,(3\sin\theta - 1) = 0.$$

So $\cos\theta = 0$ or $\sin\theta = \frac{1}{3}$.

↪ *If you divide by* $\cos\theta$ *you lose* $\cos\theta = 0$ *and the mark for* $\frac{\pi}{2}$.

Step 5. (b) Solve cos θ equal to 0.

In $0 \le \theta \le \pi$: $\theta = \dfrac{\pi}{2}$ (about 1.571).

Step 6. (b) Solve sin θ equal to one third.

$\theta = \arcsin\frac{1}{3} = 0.3398\ldots \approx 0.340$, and, by symmetry, $\theta = \pi - 0.3398\ldots = 2.8017\ldots \approx 2.802$. Both are in $[0, \pi]$.

Step 7. (b) Check.

At $\theta = 0.340$: $3\sin 0.680 = 3(0.6285) = 1.886$ and $2\cos 0.340 = 2(0.9428) = 1.886$ ✓.

Answer. (b) $\theta = 0.340,\ \dfrac{\pi}{2},\ 2.802$

Common errors.

- Using degrees when the range is in radians.
- Dividing by $\cos\theta$ and losing $\theta = \frac{\pi}{2}$.

How the marks are earned.

M1 (a) $1 - \cos 2\theta = 2\sin^2\theta$

M1 $\sin 2\theta = 2\sin\theta\cos\theta$

A1* $\dfrac{2\sin^2\theta}{2\sin\theta\cos\theta} = \dfrac{\sin\theta}{\cos\theta} = \tan\theta$

M1 (b) $6\sin\theta\cos\theta = 2\cos\theta$

M1 $2\cos\theta(3\sin\theta - 1) = 0$ *(Factorises (no division by $\cos\theta$).)*

A1 $\theta = \dfrac{\pi}{2}$

M1 $\sin\theta = \frac{1}{3} \Rightarrow \theta = 0.340$

A1 $\theta = 0.340,\ 2.802$ *(Both, and $\frac{\pi}{2}$, no extras.)*

∎

Question 9 *Differentiation* 4 marks

> *Prove, from first principles, that the derivative of x^2 is $2x$.*

Solution.

The idea.

*First principles: the gradient of the chord from x to $x+h$, then let $h \to 0$. For (b), the tangent touches at P, so the equation "curve = tangent" has $x = 1$ as a **double** root, which makes the cubic easy to factorise.*

Step 1. The chord gradient.

Take points (x, x^2) and $(x+h, (x+h)^2)$ on $y = x^2$:

$$\text{gradient of chord} = \frac{(x+h)^2 - x^2}{h} = \frac{x^2 + 2xh + h^2 - x^2}{h} = \frac{2xh + h^2}{h} = 2x + h.$$

(Cancelling h is allowed because $h \neq 0$.)

Step 2. Take the limit.

$$\frac{dy}{dx} = \lim_{h \to 0}(2x + h) = 2x. \quad \blacksquare$$

↪ *Write "$\lim_{h \to 0}$" or "as $h \to 0$". Just writing "$= 2x$" because "$h = 0$" loses the final mark: you never substitute $h = 0$.*

Answer. Proof as above.

Common errors.

- Substituting $h = 0$ before cancelling.
- Not spotting the repeated root and trying to solve the cubic by trial and error.

How the marks are earned.

B1 $(x+h)^2 = x^2 + 2xh + h^2$

M1 $\dfrac{(x+h)^2 - x^2}{h}$

A1 $= 2x + h$

A1* As $h \to 0$, gradient $\to 2x$, with limiting notation

∎

Question 10 *Tangents to curves* 6 marks

> *The curve C has equation $y = x^3 - 4x$. The point $P(1, -3)$ lies on C.*
>
> *Find an equation of the tangent to C at P, and find the coordinates of the point where this tangent meets C again.*

Solution.

The idea.

*Find the tangent from the gradient at P. Where it meets the curve again, the cubic "curve minus tangent" has a **double** root at the point of contact, which makes it easy to factorise.*

Step 1. Gradient at P.

$$\frac{dy}{dx} = 3x^2 - 4; \qquad \text{at } x = 1: \ 3 - 4 = -1.$$

Step 2. Equation of the tangent.

Through $(1, -3)$ with gradient -1:

$$y + 3 = -(x - 1) \quad \Longrightarrow \quad y = -x - 2.$$

Step 3. Where the tangent meets the curve.

Set the two expressions for y equal:

$$x^3 - 4x = -x - 2 \quad \Longrightarrow \quad x^3 - 3x + 2 = 0.$$

Step 4. Factorise using the double root.

The tangent touches the curve at $x = 1$, so $x = 1$ is a **repeated** root and $(x - 1)^2 = x^2 - 2x + 1$ is a factor. Comparing coefficients:

$$x^3 - 3x + 2 = (x^2 - 2x + 1)(x + 2).$$

Check: $(x^2 - 2x + 1)(x + 2) = x^3 + 2x^2 - 2x^2 - 4x + x + 2 = x^3 - 3x + 2$ ✓.

Step 5. The other intersection.

$x = -2$, and on the tangent $y = -(-2) - 2 = 0$. Also on the curve: $(-8) - 4(-2) = 0$ ✓. The tangent meets C again at $(-2, 0)$.

> **Answer.** $y = -x - 2$; it meets C again at $(-2, 0)$

How the marks are earned.

M1 $\dfrac{dy}{dx} = 3x^2 - 4$
A1 Gradient at P is -1
M1 Tangent $y = -x - 2$
M1 $x^3 - 4x = -x - 2 \Rightarrow x^3 - 3x + 2 = 0$
M1 Uses factor $(x - 1)^2$: $(x - 1)^2(x + 2) = 0$
A1 $(-2, 0)$

∎

Question 11 *Integration* 7 marks

> ***(a)*** *Use integration by parts to find* $\displaystyle\int xe^{2x}\,dx$. *(4)*
>
> ***(b)*** *Hence find the exact value of* $\displaystyle\int_0^1 xe^{2x}\,dx$. *(3)*

Solution.

The idea.

Integration by parts, $\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$, *reverses the product rule. Choose* $u = x$, *because differentiating it makes it simpler. Part (c) is a "reverse chain rule": the derivative of the inside,* $2x$, *is sitting outside.*

Step 1. (a) Choose u and dv.

$$u = x \Rightarrow \frac{du}{dx} = 1, \qquad \frac{dv}{dx} = e^{2x} \Rightarrow v = \tfrac{1}{2}e^{2x}.$$

↪ *If you chose* $u = e^{2x}$ *instead, you'd have to integrate* x, *giving* $\frac{x^2}{2}$: *the new integral would be* ***harder****.*

Step 2. (a) Apply the formula.

$$\int xe^{2x}\,dx = x\cdot\tfrac{1}{2}e^{2x} - \int \tfrac{1}{2}e^{2x}\cdot 1\,dx = \tfrac{1}{2}xe^{2x} - \tfrac{1}{4}e^{2x} + c.$$

Step 3. (a) Check by differentiating.

$$\frac{d}{dx}\left(\tfrac{1}{2}xe^{2x} - \tfrac{1}{4}e^{2x}\right) = \tfrac{1}{2}e^{2x} + xe^{2x} - \tfrac{1}{2}e^{2x} = xe^{2x}.\checkmark$$

Step 4. (b) Substitute the limits.

Let $F(x) = \left(\frac{1}{2}x - \frac{1}{4}\right)e^{2x}$.

$$F(1) = \tfrac{1}{4}e^2, \qquad F(0) = -\tfrac{1}{4}e^0 = -\tfrac{1}{4}.$$

$$\int_0^1 xe^{2x}\,dx = \tfrac{1}{4}e^2 - \left(-\tfrac{1}{4}\right) = \frac{e^2+1}{4} \ (\approx 2.097).$$

↪ *Don't forget the lower limit:* $e^0 = 1$, *not* 0.

Answer. (a) $\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c$ (b) $\dfrac{e^2+1}{4}$

Common errors.

- $\int e^{2x}\,dx = 2e^{2x}$ (it is $\frac{1}{2}e^{2x}$).

How the marks are earned.

M1 **(a)** $u = x,\ \dfrac{dv}{dx} = e^{2x},\ v = \frac{1}{2}e^{2x}$ and the parts structure

A1 $\frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}\,dx$

M1 Integrates the second term

A1 $\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c$

M1 **(b)** Substitutes limits 1 and 0

M1 $\left(\frac{1}{2}e^2 - \frac{1}{4}e^2\right) - \left(0 - \frac{1}{4}\right)$

A1 $\dfrac{e^2 + 1}{4}$

∎

Question 12 *Reverse chain rule* 3 marks

Find $\displaystyle\int 6x(x^2+1)^2\,dx$.

Solution.

The idea.

The integrand is (a multiple of) the derivative of the inside of the bracket, times a power of the bracket. That is a reverse chain rule: guess $(x^2+1)^3$ *and check by differentiating.*

Step 1. Spot the reverse chain rule.

The inside of the bracket is x^2+1, with derivative $2x$, and $6x = 3\times 2x$ is outside. Try $(x^2+1)^3$:

$$\frac{d}{dx}(x^2+1)^3 = 3(x^2+1)^2\times 2x = 6x(x^2+1)^2.$$

That is exactly the integrand, so

$$\int 6x(x^2+1)^2\,dx = (x^2+1)^3 + c.$$

Step 2. The same by substitution.

Let $u = x^2+1$, so $du = 2x\,dx$ and $6x\,dx = 3\,du$:

$$\int 3u^2\,du = u^3 + c = (x^2+1)^3 + c.$$

Answer. $(x^2+1)^3 + c$

Common errors.

- Expanding $(x^2+1)^2$ first. It works, but takes longer and invites errors.
- Forgetting the $+c$.

How the marks are earned.

M1 $k(x^2+1)^3$
M1 $k=1$ found by differentiating or substitution
A1 $(x^2+1)^3+c$

∎

Question 13 *Partial fractions* 9 marks

$$f(x) = \frac{7x+1}{(1+x)(1-2x)}$$

(a) *Express $f(x)$ in partial fractions.* ***(3)***

(b) *Hence find the series expansion of $f(x)$ in ascending powers of x, up to and including the term in x^2, simplifying each term.* ***(5)***

(c) *State the range of values of x for which your expansion is valid.* ***(1)***

Solution.

The idea.

*Split the fraction into two simple ones; each simple one is a geometric-style binomial expansion. The expansion is only valid where **both** pieces are.*

Step 1. (a) Set up.

$$\frac{7x+1}{(1+x)(1-2x)} \equiv \frac{A}{1+x} + \frac{B}{1-2x}.$$

Multiply through by the denominator:

$$7x+1 \equiv A(1-2x) + B(1+x).$$

Step 2. (a) Choose x values that kill a term.

- $x=-1$: $-6 = A(3) \Rightarrow A = -2$.
- $x=\frac{1}{2}$: $\frac{7}{2}+1 = \frac{9}{2} = B\left(\frac{3}{2}\right) \Rightarrow B = 3$.

$$f(x) = -\frac{2}{1+x} + \frac{3}{1-2x}.$$

Check at $x=0$: $f(0) = 1$ and $-2+3 = 1$ ✓.

Step 3. (b) Expand each piece.

Using $(1+y)^{-1} = 1 - y + y^2 - \cdots$:

$$(1+x)^{-1} = 1 - x + x^2 - \cdots$$

$$(1-2x)^{-1} = 1 + (2x) + (2x)^2 + \cdots = 1 + 2x + 4x^2 + \cdots$$

Step 4. (b) Multiply by the constants.

$$-2(1 - x + x^2) = -2 + 2x - 2x^2, \qquad 3(1 + 2x + 4x^2) = 3 + 6x + 12x^2.$$

Step 5. (b) Add.

$$f(x) = (-2+3) + (2+6)x + (-2+12)x^2 + \cdots = 1 + 8x + 10x^2 + \cdots$$

Step 6. (b) Check by multiplying out.

$(1+x)(1-2x) = 1 - x - 2x^2$, and

$$(1 - x - 2x^2)(1 + 8x + 10x^2) = 1 + 8x + 10x^2 - x - 8x^2 - 2x^2 + \cdots = 1 + 7x + 0x^2 + \cdots \checkmark$$

which matches the numerator $1 + 7x$.

Step 7. (c) Validity.

$(1+x)^{-1}$ needs $|x| < 1$; $(1-2x)^{-1}$ needs $|2x| < 1$, i.e. $|x| < \frac{1}{2}$. Both must hold, so the stricter one wins:

$$|x| < \tfrac{1}{2}.$$

Answer. (a) $-\dfrac{2}{1+x} + \dfrac{3}{1-2x}$ (b) $1 + 8x + 10x^2$ (c) $|x| < \frac{1}{2}$

Common errors.

- $(1-2x)^{-1} = 1 + 2x + 2x^2$ (the $(2x)^2$ trap again).
- Giving $|x| < 1$ for validity.

How the marks are earned.

M1 (a) $\dfrac{A}{1+x} + \dfrac{B}{1-2x}$ and $7x + 1 = A(1-2x) + B(1+x)$
M1 Substitutes values or compares coefficients
A1 $A = -2$, $B = 3$
M1 (b) $(1+x)^{-1} = 1 - x + x^2$
M1 $(1-2x)^{-1} = 1 + 2x + 4x^2$
A1 One expansion correct and multiplied by their constant
M1 Adds the two series
A1 $1 + 8x + 10x^2$
B1 (c) $|x| < \frac{1}{2}$

∎

Question 14 *Parametric equations* 6 marks

The curve C has parametric equations

$$x = t^2, \qquad y = t^3 - 3t, \qquad t \in \mathbb{R}.$$

(a) *Find $\dfrac{dy}{dx}$ in terms of t.* *(3)*

(b) *Find an equation of the tangent to C at the point where $t = 2$, in the form $ax + by + c = 0$.* *(3)*

Solution.

The idea.

*Parametric curves are traced as t runs: watch the video to see this one draw its loop. Derivatives come from the chain rule, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$. A self-crossing means two **different** t values give the **same** point. The loop is symmetric in the x-axis, so its area is twice the area of the top half.*

Step 1. (a) Differentiate each with respect to t.

$$\frac{dx}{dt} = 2t, \qquad \frac{dy}{dt} = 3t^2 - 3.$$

Step 2. (a) Divide.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 3}{2t}.$$

Step 3. (b) The point and the gradient.

At $t = 2$: $x = 4$, $y = 8 - 6 = 2$, so the point is $(4, 2)$. The gradient is $\dfrac{12 - 3}{4} = \dfrac{9}{4}$.

Step 4. (b) The tangent.

$$y - 2 = \tfrac{9}{4}(x - 4) \quad \Longrightarrow \quad 4y - 8 = 9x - 36 \quad \Longrightarrow \quad 9x - 4y - 28 = 0.$$

Check $(4, 2)$: $36 - 8 - 28 = 0$ ✓.

Answer. (a) $\dfrac{3t^2 - 3}{2t}$ (b) $9x - 4y - 28 = 0$

How the marks are earned.

M1 (a) $\dfrac{dx}{dt} = 2t$, $\dfrac{dy}{dt} = 3t^2 - 3$
M1 $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$
A1 $\dfrac{3t^2 - 3}{2t}$
M1 (b) Point $(4, 2)$ and gradient $\frac{9}{4}$
M1 $y - 2 = \frac{9}{4}(x - 4)$
A1 $9x - 4y - 28 = 0$

∎

Question 15 *Parametric curves and area* 11 marks

The curve C has parametric equations

$$x = t^2, \qquad y = t^3 - 3t, \qquad t \in \mathbb{R}.$$

(a) *Show that C crosses itself at the point $(3, 0)$.* ***(2)***

(b) *Show that a Cartesian equation of C is $y^2 = x(x-3)^2$.* ***(3)***

The curve C forms a closed loop between $x = 0$ and $x = 3$.

(c) *Find the exact area enclosed by the loop.* ***(6)***

A video explanation of this question is at lemmamathematics.com/papers/pure-2#q15.

Solution.

The idea.
*The video shows this curve tracing its loop as t increases. A self-crossing means two **different** values of t give the **same** point. The loop is symmetric about the x-axis, so its area is twice the area of its top half.*

Step 1. (a) Which t values give (3, 0).
$x = 3$ needs $t^2 = 3$, so $t = \pm\sqrt{3}$. Then

$$y = t(t^2 - 3) = t(3 - 3) = 0 \quad \text{for both.}$$

So $t = \sqrt{3}$ and $t = -\sqrt{3}$, two different parameter values, both arrive at $(3, 0)$: the curve passes through that point twice, i.e. it crosses itself.

Step 2. (b) Eliminate t.
Factorise $y = t(t^2 - 3)$. Squaring removes the sign of t, which lets me use $t^2 = x$:

$$y^2 = t^2(t^2 - 3)^2 = x(x - 3)^2. \quad \blacksquare$$

Step 3. (c) Use the symmetry.
$y^2 = x(x-3)^2$ is unchanged when $y \to -y$, so the loop is symmetric about the x-axis. For $0 \le x \le 3$ the top half is

$$y = \sqrt{x}\,|x - 3| = \sqrt{x}\,(3 - x).$$

$$\text{Area} = 2\int_0^3 \sqrt{x}\,(3 - x)\,dx.$$

$\hookrightarrow |x - 3| = 3 - x$ *because $x \le 3$ inside the loop.*

Step 4. (c) Expand and integrate.

$$\sqrt{x}(3 - x) = 3x^{\frac{1}{2}} - x^{\frac{3}{2}}.$$

$$\int (3x^{\frac{1}{2}} - x^{\frac{3}{2}})\,dx = 3\cdot\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{5}{2}}}{\frac{5}{2}} = 2x^{\frac{3}{2}} - \tfrac{2}{5}x^{\frac{5}{2}}.$$

Step 5. (c) Substitute the limits.
$3^{\frac{3}{2}} = 3\sqrt{3}$ and $3^{\frac{5}{2}} = 9\sqrt{3}$:

$$\left[2x^{\frac{3}{2}} - \tfrac{2}{5}x^{\frac{5}{2}}\right]_0^3 = 6\sqrt{3} - \tfrac{18}{5}\sqrt{3} = \tfrac{12}{5}\sqrt{3}.$$

Step 6. (c) Double it.

$$\text{Area} = 2\times\frac{12\sqrt{3}}{5} = \frac{24\sqrt{3}}{5} \approx 8.31.$$

Step 7. (c) Check using t.
The loop is traced once as t goes from $-\sqrt{3}$ to $\sqrt{3}$, so the area is $\left|\int y\,\frac{dx}{dt}\,dt\right|$:

$$\int_{-\sqrt{3}}^{\sqrt{3}} (t^3 - 3t)(2t)\,dt = \left[\tfrac{2}{5}t^5 - 2t^3\right]_{-\sqrt{3}}^{\sqrt{3}} = 2\left(\tfrac{18\sqrt{3}}{5} - 6\sqrt{3}\right) = -\frac{24\sqrt{3}}{5}.$$

The minus sign only records the direction the loop is traced (clockwise). The size, $\frac{24\sqrt{3}}{5}$, agrees. ✓

Answer. (c) $\dfrac{24\sqrt{3}}{5}$

Common errors.

- Forgetting to double in (c), giving $\frac{12\sqrt{3}}{5}$.
- Writing the top half as $\sqrt{x}(x-3)$, which is negative and gives a negative "area".

Examiner's note.

When a parametric area comes out negative, don't panic: take the modulus and say it reflects the direction of travel.

How the marks are earned.

M1 (a) $t=\sqrt{3}$ and $t=-\sqrt{3}$ both give $(3,0)$
A1 Two different values of t give the same point, so the curve crosses itself
M1 (b) $y = t(t^2-3)$
M1 $y^2 = t^2(t^2-3)^2$
A1* $= x(x-3)^2$
M1 (c) Area $= 2\int_0^3 \sqrt{x}(3-x)\,dx$ or $\left|\int_{-\sqrt{3}}^{\sqrt{3}} y\frac{dx}{dt}\,dt\right|$
M1 $3x^{\frac{1}{2}} - x^{\frac{3}{2}}$ or $2t^4 - 6t^2$
M1 Integrates: $2x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}}$ or $\frac{2}{5}t^5 - 2t^3$
A1 Correct integral
dM1 Correct limits and doubling (or modulus)
A1 $\dfrac{24\sqrt{3}}{5}$

∎