Name _____ Date _____ Score _____ / 100

LEMMA PRACTICE PAPER

Level 3

Pure Mathematics

Paper 3 · Challenging exam

Paper reference **LP3/P** Time **2 hours**

TOTAL MARKS	100

You must have: pen, pencil, ruler, graph paper and a calculator.

INSTRUCTIONS

- Use black ink or ball-point pen; pencil may be used for diagrams.
- Answer **all** questions, in the spaces provided. If you need more space, continue on extra paper and label it clearly.
- Show all your working. An answer with no supporting method may get no marks.
- Give answers to 3 significant figures unless a question says otherwise, and give exact answers when asked.

INFORMATION

- There are **15** questions, and the paper is out of **100** marks.
- The marks for each part of a question are shown in brackets, for example **(3)**. Use them to judge how much to write.
- The questions are in roughly increasing order of difficulty.
- Difficulty: **Challenging exam**. The harder end of a real exam paper, but still realistic: longer chains of reasoning, less signposting and a demanding final question. This is the level to aim at for top grades.

ADVICE

- Read each question carefully before you start.
- Keep an eye on the time: roughly one minute per mark.
- Check your answers if you have time at the end.
- The mark scheme, very detailed model solutions and video explanations are free at **lemmamathematics.com/papers/pure-3**.

1 The equation **(5)**

$$(k+3)x^2 + 4x + k = 0,$$

where k is a constant, has no real roots. Find the set of possible values of k.

(Total for Question 1 is 5 marks)

2 Prove by contradiction that $\log_2 3$ is irrational. **(5)**

(Total for Question 2 is 5 marks)

3 In the binomial expansion of $(1 + ax)^n$, where n is a positive integer and a is a constant, the coefficient of x is 12 and the coefficient of x^2 is 60.

(a) Find the value of n and the value of a. **(5)**

(b) Find the coefficient of x^3. **(2)**

(Total for Question 3 is 7 marks)

4 **(a)** On the same axes, sketch the graphs of $y = |2x - 3|$ and $y = 5 - x$, showing the coordinates of the points where each graph meets the axes. **(2)**

(b) Solve $|2x - 3| = 5 - x$. **(3)**

(c) Hence, or otherwise, solve $|2x - 3| > 5 - x$. **(2)**

(Total for Question 4 is 7 marks)

5 **(a)** Express $3\sin\theta - 4\cos\theta$ in the form $R\sin(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. Give the value of α to 4 decimal places. **(3)**

(b) Hence solve, for $0 \leq \theta < 2\pi$, **(3)**

$$3\sin\theta - 4\cos\theta = 2.5,$$

giving your answers to 3 decimal places.

(Total for Question 5 is 6 marks)

6 It is given that $3\sin\theta - 4\cos\theta \equiv 5\sin(\theta - \alpha)$, where $\alpha = 0.9273$ to 4 decimal places. **(3)**

Find the maximum value of

$$\frac{10}{7 + 3\sin\theta - 4\cos\theta}$$

and the smallest positive value of θ at which it occurs.

(Total for Question 6 is 3 marks)

7 Two variables x and y are related by $y = ab^x$, where a and b are constants.

(a) Show that the graph of $\ln y$ against x is a straight line. **(2)**

The graph of $\ln y$ against x passes through the points $(0, 1.2)$ and $(5, 3.7)$.

(b) Find the values of a and b, giving your answers to 3 significant figures. **(4)**

(c) Use the model to estimate y when $x = 8$, and comment on the reliability of your estimate given that the data were collected for $0 \leq x \leq 5$. **(3)**

(Total for Question 7 is 9 marks)

8 A spherical balloon is being inflated. Its volume is increasing at a constant rate of $20\ \text{cm}^3\,\text{s}^{-1}$. [The volume of a sphere is $V = \frac{4}{3}\pi r^3$ and its surface area is $S = 4\pi r^2$.]

(a) Find the rate at which the radius is increasing when the radius is 5 cm, giving your answer to 3 significant figures. **(4)**

(b) Show that, at any time, $\dfrac{dS}{dt} = \dfrac{40}{r}$. **(3)**

(c) Find the radius of the balloon at the instant when its surface area is increasing at $2\ \text{cm}^2\,\text{s}^{-1}$. **(2)**

(Total for Question 8 is 9 marks)

9 Use the substitution $u = \sqrt{x}$ to show that **(6)**

$$\int_1^4 \frac{1}{x + \sqrt{x}}\,dx = \ln\frac{9}{4}.$$

(Total for Question 9 is 6 marks)

10 Find the exact value of $\int_0^{\frac{\pi}{2}} x \sin x \, dx$. **(4)**

(Total for Question 10 is 4 marks)

11 Relative to a fixed origin O, the points A, B and C have position vectors

$$\mathbf{a} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}, \qquad \mathbf{b} = 4\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}, \qquad \mathbf{c} = 3\mathbf{j} + 5\mathbf{k}.$$

(a) Find $|\overrightarrow{AB}|$. **(2)**

(b) Find the size of angle BAC, in degrees to one decimal place. **(3)**

(c) Find the exact area of triangle ABC. **(2)**

(Total for Question 11 is 7 marks)

12 Relative to a fixed origin O, the points A, B and C have position vectors

$$\mathbf{a} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}, \qquad \mathbf{b} = 4\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}, \qquad \mathbf{c} = 3\mathbf{j} + 5\mathbf{k}.$$

You are given that $|\overrightarrow{AB}| = 6$ and that the area of triangle ABC is $10\sqrt{2}$.

(a) Find the position vector of the point D such that $ABCD$ is a parallelogram. **(2)**

(b) Find the exact shortest distance from C to the line through A and B. **(2)**

(Total for Question 12 is 4 marks)

13 Water drains from a tank. The depth of the water, h metres, t minutes after the water starts to drain, satisfies

$$\frac{dh}{dt} = -k\sqrt{h},$$

where k is a positive constant. Initially the depth is 16 m, and after 10 minutes it is 9 m.

(a) Show that $2\sqrt{h} = 8 - 0.2t$. **(5)**

(b) Find the time taken for the tank to empty. **(2)**

(c) Find the time taken for the depth to fall from 16 m to 4 m. **(2)**

(d) Give one criticism of the model. **(1)**

(Total for Question 13 is 10 marks)

14 The curve C has equation

$$y = (x^2 - 3)e^{-x}.$$

(a) Show that $\dfrac{dy}{dx} = -(x - 3)(x + 1)e^{-x}$. **(3)**

(b) Find the exact coordinates of the stationary points of C, and determine their nature. **(4)**

(Total for Question 14 is 7 marks)

15 The curve C has equation $y = (x^2 - 3)e^{-x}$. It has a minimum point at $(-1, -2e)$ and a maximum point at $(3, 6e^{-3})$.

(a) Find the exact area of the finite region bounded by C and the x-axis. **(8)**

(b) Find the set of values of k for which the line $y = k$ meets C at exactly three points. **(3)**

(Total for Question 15 is 11 marks)

TOTAL FOR PAPER IS 100 MARKS

END OF PAPER