Pure Mathematics

Paper 3 · Challenging exam: model solutions

LP3/P · Level 3

How to use these solutions. Try each question under timed conditions first, then mark it with the mark scheme. Read the full solution for every question, even the ones you got right: they often show a faster method, a check, or a common error to avoid. Each solution begins with *the idea*, then works through numbered steps; remarks marked ↪ explain *why*. Each ends with the boxed answer and how the marks are earned.

Question 1 *Discriminant* 5 marks

> *The equation*
>
> $$(k+3)x^2 + 4x + k = 0,$$
>
> *where* k *is a constant, has no real roots. Find the set of possible values of* k*.*

Solution.

The idea.

"No real roots" means the discriminant is negative. That gives a quadratic ***inequality in*** k*. Be careful with the direction when dividing by a negative.*

Step 1. Identify a, b and c.

$$a = k+3, \qquad b = 4, \qquad c = k.$$

↪ *The coefficients can contain* k*; that is the whole point of the question.*

Step 2. Write the condition.

$$b^2 - 4ac < 0 \quad \Longrightarrow \quad 16 - 4k(k+3) < 0.$$

Step 3. Simplify.

$$16 - 4k^2 - 12k < 0.$$

Divide by -4, which **reverses** the inequality:

$$k^2 + 3k - 4 > 0.$$

Step 4. Critical values.

$$k^2 + 3k - 4 = (k+4)(k-1) = 0 \quad \Longrightarrow \quad k = -4 \text{ or } k = 1.$$

Step 5. Choose the region.
$k^2 + 3k - 4$ is a U-shaped quadratic in k, positive **outside** its roots:

$$k < -4 \quad \text{or} \quad k > 1.$$

Step 6. Check the edge case.
If $k = -3$ the equation stops being a quadratic ($4x - 3 = 0$, which **does** have a root). But $k = -3$ is not in my answer anyway, so no adjustment is needed. **Test $k = 2$:** $5x^2 + 4x + 2$ has discriminant $16 - 40 < 0$ ✓. **Test $k = 0$:** $3x^2 + 4x$ has roots ✓ (correctly excluded).

Answer. $k < -4$ or $k > 1$

Common errors.

- Not reversing the inequality when dividing by -4, giving $-4 < k < 1$.
- Using $b^2 - 4ac \leq 0$ (equal roots are still real roots).

How the marks are earned.
M1 Uses $b^2 - 4ac < 0$
A1 $16 - 4k(k+3) < 0$
A1 $k^2 + 3k - 4 > 0$
M1 Critical values $k = -4,\ k = 1$ and chooses the outside region
A1 $k < -4$ or $k > 1$

∎

Question 2 *Proof by contradiction* 5 marks

> *Prove by contradiction that* $\log_2 3$ *is irrational.*

Solution.

The idea.

To prove something is irrational, assume it is rational, $\frac{p}{q}$*, and show this leads to something impossible. Here the impossibility is about* ***odd and even****.*

Step 1. Make the assumption.

Assume, for contradiction, that $\log_2 3$ is rational. Since $\log_2 3 > 0$ (because $3 > 1$), I can write

$$\log_2 3 = \frac{p}{q}, \qquad p, q \text{ positive integers.}$$

Step 2. Remove the logarithm.

$\log_2 3 = \frac{p}{q}$ means $2^{p/q} = 3$. Raise both sides to the power q:

$$2^p = 3^q.$$

Step 3. Look at parity.

- 2^p is a product of $p \geq 1$ twos, so it is **even**.
- 3^q is a product of odd numbers, so it is **odd**.

Step 4. The contradiction.

An even number cannot equal an odd number. So $2^p = 3^q$ is impossible, and the assumption was false. Therefore $\log_2 3$ is irrational. ■

↪ *I needed* $p \geq 1$ *for* 2^p *to be even (*$2^0 = 1$ *is odd). That's why I noted* $\log_2 3 > 0$ *at the start.*

Answer. Proof as above.

Common errors.

- Not stating the assumption clearly at the start.
- Ending without a conclusion: say what the contradiction proves.

How the marks are earned.

B1 Assumes $\log_2 3 = \frac{p}{q}$ with p, q positive integers *(Must say "assume, for contradiction".)*

M1 $2^{\frac{p}{q}} = 3 \Rightarrow 2^p = 3^q$

M1 Notes 2^p is even (as $p \geq 1$) and 3^q is odd

A1 Identifies the contradiction: an even number cannot equal an odd number

A1 Concludes that $\log_2 3$ is irrational

Question 3 *Binomial expansion* 7 marks

In the binomial expansion of $(1+ax)^n$, where n is a positive integer and a is a constant, the coefficient of x is 12 and the coefficient of x^2 is 60.

(a) *Find the value of n and the value of a.* ***(5)***

(b) *Find the coefficient of x^3.* ***(2)***

Solution.

The idea.

Write down the general coefficients, $\binom{n}{1}a$ and $\binom{n}{2}a^2$, set up two equations and eliminate a cleverly by dividing.

Step 1. (a) The two coefficients.

$$(1+ax)^n = 1 + n(ax) + \frac{n(n-1)}{2}(ax)^2 + \cdots$$

So

$$na = 12 \quad (1), \qquad \frac{n(n-1)}{2}a^2 = 60 \quad (2).$$

Step 2. (a) Eliminate a.

From (2), $n(n-1)a^2 = 120$. Squaring (1) gives $n^2a^2 = 144$. Divide:

$$\frac{n(n-1)a^2}{n^2a^2} = \frac{120}{144} \quad \Longrightarrow \quad \frac{n-1}{n} = \frac{5}{6}.$$

↪ *Dividing like this kills a^2 completely: much faster than substituting $a = \frac{12}{n}$, though that works too.*

Step 3. (a) Solve.

$6(n-1) = 5n \Rightarrow 6n - 6 = 5n \Rightarrow n = 6$. Then $a = \dfrac{12}{6} = 2$. **Check:** $\binom{6}{2} \times 2^2 = 15 \times 4 = 60$ ✓.

Step 4. (b) The x cubed term.

$$\binom{6}{3}(2x)^3 = 20 \times 8x^3 = 160x^3.$$

The coefficient is 160.

Answer. (a) $n = 6$, $a = 2$ (b) 160

Common errors.

- Writing the x^2 coefficient as $\frac{n(n-1)}{2}a$ (forgetting to square a).

How the marks are earned.

M1 (a) $na = 12$

M1 $\dfrac{n(n-1)}{2}a^2 = 60$

M1 Eliminates a, e.g. $\dfrac{n-1}{n} = \dfrac{120}{144}$

A1 $n = 6$

A1 $a = 2$

M1 (b) $\binom{6}{3}a^3$

A1 160

∎

Question 4 *Modulus* 7 marks

> ***(a)*** *On the same axes, sketch the graphs of* $y = |2x - 3|$ *and* $y = 5 - x$*, showing the coordinates of the points where each graph meets the axes.* *(2)*
>
> ***(b)*** *Solve* $|2x - 3| = 5 - x$*.* *(3)*
>
> ***(c)*** *Hence, or otherwise, solve* $|2x - 3| > 5 - x$*.* *(2)*

Solution.

The idea.

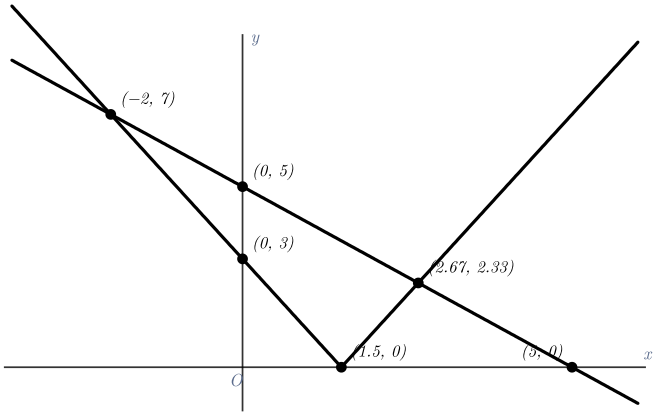
$|2x - 3|$ *is a V shape: the line* $y = 2x - 3$ *with its negative part flipped up. Solve each "arm" separately, check against the graph, then read the inequality from the sketch.*

Step 1. (a) The V shape.

$|2x - 3| = 0$ when $x = \frac{3}{2}$: that's the vertex $\left(\frac{3}{2}, 0\right)$. At $x = 0$, $|-3| = 3$, so it crosses the y-axis at $(0, 3)$. The right arm has gradient 2, the left arm gradient -2.

Step 2. (a) The line.

$y = 5 - x$ crosses the axes at $(0, 5)$ and $(5, 0)$.



Step 3. (b) The right arm.

Where $2x - 3 \geq 0$, $|2x - 3| = 2x - 3$:

$$2x - 3 = 5 - x \quad \Longrightarrow \quad 3x = 8 \quad \Longrightarrow \quad x = \frac{8}{3}.$$

Valid, since $\frac{8}{3} > \frac{3}{2}$.

Step 4. (b) The left arm.

Where $2x - 3 < 0$, $|2x - 3| = -(2x - 3) = 3 - 2x$:

$$3 - 2x = 5 - x \quad \Longrightarrow \quad -x = 2 \quad \Longrightarrow \quad x = -2.$$

Valid, since $-2 < \frac{3}{2}$. **Check:** $|-7| = 7 = 5 - (-2)$ ✓; $\left|\frac{7}{3}\right| = \frac{7}{3} = 5 - \frac{8}{3}$ ✓.

Step 5. (c) Read from the sketch.

$|2x-3| > 5-x$ means the V is **above** the line. The line is above the V between the two crossing points, so the V is above outside them:

$$x < -2 \quad \text{or} \quad x > \tfrac{8}{3}.$$

Answer. (b) $x = -2$ or $x = \frac{8}{3}$ (c) $x < -2$ or $x > \frac{8}{3}$

Common errors.

- Only solving $2x - 3 = 5 - x$.
- Squaring both sides without checking for invalid roots (it works here, but needs care).

How the marks are earned.

B1 **(a)** V shape with vertex $\left(\frac{3}{2}, 0\right)$ and y-intercept $(0, 3)$
B1 Line with intercepts $(0, 5)$ and $(5, 0)$
M1 **(b)** Solves $2x - 3 = 5 - x$ or $-(2x - 3) = 5 - x$
A1 $x = \frac{8}{3}$
A1 $x = -2$
M1 **(c)** Chooses the outside region from their critical values
A1 $x < -2$ or $x > \frac{8}{3}$

∎

Question 5 *Harmonic form* 6 marks

(a) *Express* $3\sin\theta - 4\cos\theta$ *in the form* $R\sin(\theta - \alpha)$*, where* $R > 0$ *and* $0 < \alpha < \dfrac{\pi}{2}$*. Give the value of* α *to 4 decimal places.* *(3)*

(b) *Hence solve, for* $0 \le \theta < 2\pi$*,* *(3)*

$$3\sin\theta - 4\cos\theta = 2.5,$$

giving your answers to 3 decimal places.

Solution.

The idea.

Any combination $a\sin\theta + b\cos\theta$ *is a single sine wave* $R\sin(\theta - \alpha)$*. That turns hard equations into easy ones, and makes maxima obvious, because* $\sin$ *lies between* -1 *and* 1*.*

Step 1. (a) Expand the target form.

$$R\sin(\theta - \alpha) = R\sin\theta\cos\alpha - R\cos\theta\sin\alpha.$$

Match coefficients with $3\sin\theta - 4\cos\theta$:

$$R\cos\alpha = 3, \qquad R\sin\alpha = 4.$$

Step 2. (a) Find R and alpha.

Square and add (using $\cos^2 + \sin^2 = 1$):

$$R^2 = 3^2 + 4^2 = 25 \quad \Longrightarrow \quad R = 5.$$

Divide: $\tan\alpha = \frac{4}{3}$, so $\alpha = \arctan\frac{4}{3} = 0.9273$ (4 d.p.).

$$3\sin\theta - 4\cos\theta = 5\sin(\theta - 0.9273).$$

Step 3. (b) Rewrite the equation.

$$5\sin(\theta - \alpha) = 2.5 \quad \Longrightarrow \quad \sin(\theta - \alpha) = \tfrac{1}{2}.$$

Step 4. (b) Adjust the range.

If $0 \le \theta < 2\pi$, then $-\alpha \le \theta - \alpha < 2\pi - \alpha$, roughly $-0.927 \le \theta - \alpha < 5.356$. In that interval, $\sin(\cdot) = \frac{1}{2}$ at

$$\theta - \alpha = \frac{\pi}{6} \quad \text{or} \quad \frac{5\pi}{6}.$$

(The next ones, $\frac{\pi}{6} + 2\pi$ and $-\frac{7\pi}{6}$, are outside the interval.)

Step 5. (b) Add alpha back.

$$\theta = 0.9273 + 0.5236 = 1.451, \qquad \theta = 0.9273 + 2.6180 = 3.545.$$

Check $\theta = 1.451$: $3\sin 1.451 - 4\cos 1.451 = 2.9893 - 0.4893 = 2.50$ ✓.

> **Answer.** (a) $5\sin(\theta - 0.9273)$ (b) $\theta = 1.451,\ 3.545$

Common errors.

- $\tan\alpha = \frac{3}{4}$ (dividing the wrong way round).

How the marks are earned.

B1 (a) $R = 5$
M1 $\tan\alpha = \frac{4}{3}$
A1 $\alpha = 0.9273$
M1 (b) $\sin(\theta - \alpha) = \frac{1}{2}$
M1 $\theta - \alpha = \frac{\pi}{6}$ or $\frac{5\pi}{6}$
A1 $\theta = 1.451,\ 3.545$

∎

Question 6 *Harmonic form: maxima* 3 marks

> *It is given that* $3\sin\theta - 4\cos\theta \equiv 5\sin(\theta - \alpha)$*, where* $\alpha = 0.9273$ *to 4 decimal places.*
>
> *Find the maximum value of*
>
> $$\frac{10}{7 + 3\sin\theta - 4\cos\theta}$$
>
> *and the smallest positive value of* θ *at which it occurs.*

Solution.

The idea.

With the expression written as a single sine wave, the denominator ranges between $7 - 5$ *and* $7 + 5$*. A fraction with a positive numerator is largest when its denominator is* ***smallest****, which happens when the sine equals* -1*.*

Step 1. Maximise by minimising the denominator.

$$\frac{10}{7 + 5\sin(\theta - \alpha)}.$$

A fraction with a positive top is largest when its denominator is smallest. Since $-1 \leq \sin \leq 1$, the denominator ranges from $7 - 5 = 2$ to 12; it's always positive, so there's no division by zero. The maximum value is

$$\frac{10}{2} = 5.$$

Step 2. Where it happens.

I need $\sin(\theta - \alpha) = -1$, i.e. $\theta - \alpha = \frac{3\pi}{2}$ (or $-\frac{\pi}{2}$, but that gives a negative θ):

$$\theta = 0.9273 + 4.7124 = 5.640.$$

Answer. Maximum value 5, at $\theta = 5.640$

Common errors.

- Using $\sin(\theta - \alpha) = 1$: that *maximises* the denominator and gives the minimum value, $\frac{10}{12}$.
- Giving a negative value of θ when the question asks for the smallest **positive** one.

How the marks are earned.

M1 Denominator minimum $7 - 5 = 2$
A1 Maximum value 5
A1 $\theta = \alpha + \frac{3\pi}{2} = 5.640$



Question 7 *Logarithmic models* 9 marks

Two variables x and y are related by $y = ab^x$, where a and b are constants.

(a) *Show that the graph of $\ln y$ against x is a straight line.* *(2)*

The graph of $\ln y$ against x passes through the points $(0, 1.2)$ and $(5, 3.7)$.

(b) *Find the values of a and b, giving your answers to 3 significant figures.* *(4)*

(c) *Use the model to estimate y when $x = 8$, and comment on the reliability of your estimate given that the data were collected for $0 \leq x \leq 5$.* *(3)*

Solution.

The idea.

Taking logs turns a power relationship into a straight line. Then the gradient and intercept of the line hand you $\ln b$ and $\ln a$.

Step 1. (a) Take ln of both sides.

$$\ln y = \ln(ab^x) = \ln a + \ln(b^x) = \ln a + x \ln b.$$

Step 2. (a) Compare with a straight line.

Writing $Y = \ln y$:

$$Y = (\ln b)\, x + \ln a,$$

which is $Y = mx + c$ with constant gradient $m = \ln b$ and intercept $c = \ln a$. So the graph of $\ln y$ against x is a straight line. ■

Step 3. (b) The intercept gives a.

The line passes through $(0, 1.2)$, so the intercept is $\ln a = 1.2$:

$$a = e^{1.2} = 3.3201\ldots \approx 3.32.$$

Step 4. (b) The gradient gives b.

$$\ln b = \frac{3.7 - 1.2}{5 - 0} = 0.5 \quad \Longrightarrow \quad b = e^{0.5} = 1.6487\ldots \approx 1.65.$$

Step 5. (c) Estimate.

On the line: $\ln y = 1.2 + 0.5 \times 8 = 5.2$, so

$$y = e^{5.2} = 181.27\ldots \approx 181.$$

(Using the rounded a, b: $3.32 \times 1.65^8 = 182$; either is fine.)

Step 6. (c) Comment.

$x = 8$ is outside the range of the data ($0 \le x \le 5$). This is **extrapolation**: there's no evidence the relationship continues beyond $x = 5$, so the estimate may be unreliable.

↪ *Use the word "extrapolation". Generic comments like "it's only a model" don't earn the mark.*

Answer. (b) $a = 3.32$, $b = 1.65$ (c) $y \approx 181$; unreliable, since it is extrapolation

Common errors.

- $\ln(ab^x) = \ln a \times x \ln b$.
- Using $a = 1.2$ (forgetting it is $\ln a$).

How the marks are earned.

M1 (a) $\ln y = \ln a + x \ln b$
A1 Compares with $Y = mX + c$: gradient $\ln b$, intercept $\ln a$
M1 (b) $\ln a = 1.2$
A1 $a = 3.32$
M1 $\ln b = \dfrac{3.7 - 1.2}{5} = 0.5$
A1 $b = 1.65$
M1 (c) $\ln y = 1.2 + 0.5 \times 8$
A1 $y = 181$
B1 Extrapolation, so it may be unreliable

∎

Question 8 *Connected rates of change* 9 marks

A spherical balloon is being inflated. Its volume is increasing at a constant rate of $20\ \text{cm}^3\,\text{s}^{-1}$. *[The volume of a sphere is* $V = \frac{4}{3}\pi r^3$ *and its surface area is* $S = 4\pi r^2$.*]*

(a) *Find the rate at which the radius is increasing when the radius is 5 cm, giving your answer to 3 significant figures.* *(4)*

(b) *Show that, at any time,* $\dfrac{dS}{dt} = \dfrac{40}{r}$. *(3)*

(c) *Find the radius of the balloon at the instant when its surface area is increasing at* $2\ \text{cm}^2\,\text{s}^{-1}$. *(2)*

Solution.

The idea.

Every rate links to every other through the chain rule, via r*:* $\frac{dr}{dt} = \frac{dV/dt}{dV/dr}$ *and* $\frac{dS}{dt} = \frac{dS}{dr} \cdot \frac{dr}{dt}$*. Write down what you know and what you want, as derivatives, before doing anything.*

Step 1. (a) What I know and want.

Know: $\dfrac{dV}{dt} = 20$. Want: $\dfrac{dr}{dt}$ when $r = 5$. Link: $V = \frac{4}{3}\pi r^3$, so

$$\frac{dV}{dr} = 4\pi r^2.$$

Step 2. (a) Chain rule.

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt} \quad \Longrightarrow \quad \frac{dr}{dt} = \frac{dV/dt}{dV/dr} = \frac{20}{4\pi r^2} = \frac{5}{\pi r^2}.$$

Step 3. (a) Substitute r equal to 5.

$$\frac{dr}{dt} = \frac{5}{25\pi} = \frac{1}{5\pi} = 0.06366\ldots \approx 0.0637\ \text{cm}\,\text{s}^{-1}.$$

↪ *Units check: cm³ per second divided by cm² gives cm per second ✓.*

Step 4. (b) Surface area rate.

$$\frac{dS}{dr} = 8\pi r, \qquad \frac{dS}{dt} = \frac{dS}{dr} \times \frac{dr}{dt} = 8\pi r \times \frac{5}{\pi r^2} = \frac{40}{r}. \quad \blacksquare$$

Step 5. (c) Solve for r.

$$\frac{40}{r} = 2 \quad \Longrightarrow \quad r = 20\ \text{cm}.$$

↪ *Interesting: as the balloon grows, its surface area grows more and more slowly, even though air goes in at a constant rate.*

Answer. (a) 0.0637 cm s^{-1} (c) $r = 20$ cm

Common errors.

- Multiplying by $\frac{dV}{dr}$ instead of dividing.
- Substituting $r = 5$ too early in (b), so the general result can't be shown.

How the marks are earned.

B1 (a) $\dfrac{dV}{dr} = 4\pi r^2$

M1 $\dfrac{dr}{dt} = \dfrac{dV}{dt} \div \dfrac{dV}{dr}$

M1 $\dfrac{20}{4\pi \times 25}$

A1 $0.0637\ \text{cm}\,\text{s}^{-1}$

B1 (b) $\dfrac{dS}{dr} = 8\pi r$

M1 $\dfrac{dS}{dt} = 8\pi r \times \dfrac{20}{4\pi r^2}$

A1* $= \dfrac{40}{r}$

M1 (c) $\dfrac{40}{r} = 2$

A1 $r = 20$ cm

∎

Question 9 *Integration* 6 marks

Use the substitution $u = \sqrt{x}$ to show that

$$\int_1^4 \frac{1}{x + \sqrt{x}}\,dx = \ln\frac{9}{4}.$$

Solution.

The idea.

*In (a), the substitution removes the square root, and cancelling a factor of u leaves something that integrates to a logarithm. Remember to change **all three** things: the integrand, the dx and the limits. In (b), integrate by parts, differentiating the x.*

Step 1. Express x and dx in terms of u.

$u = \sqrt{x}$ means $x = u^2$, so $\dfrac{dx}{du} = 2u$, i.e. $dx = 2u\,du$.

Step 2. Change the limits.

$x = 1 \Rightarrow u = 1$; $x = 4 \Rightarrow u = 2$.

Step 3. Rewrite the integral.

$$\int_1^4 \frac{1}{x + \sqrt{x}}\,dx = \int_1^2 \frac{1}{u^2 + u} \cdot 2u\,du = \int_1^2 \frac{2u}{u(u+1)}\,du = \int_1^2 \frac{2}{u+1}\,du.$$

Step 4. Integrate.

$$\Big[2\ln(u+1)\Big]_1^2 = 2\ln 3 - 2\ln 2 = 2\ln\frac{3}{2} = \ln\left(\frac{3}{2}\right)^2 = \ln\frac{9}{4}. \quad \blacksquare$$

(Numerically 0.811.)

Answer. $\ln\dfrac{9}{4}$ (shown)

Common errors.

- Leaving the limits as 1 and 4 after substituting.

How the marks are earned.

M1 $x = u^2,\ dx = 2u\,du$
M1 $\int \dfrac{2u}{u^2+u}\,du$
A1 $\int \dfrac{2}{u+1}\,du$
M1 $2\ln(u+1)$
M1 Correct new limits $u = 1,\ 2$
A1* $2\ln 3 - 2\ln 2 = \ln\dfrac{9}{4}$

Question 10 *Integration by parts* 4 marks

Find the exact value of $\displaystyle\int_0^{\frac{\pi}{2}} x\sin x\,dx$.

Solution.

The idea.

A product of x and a trig function: integrate by parts with $u = x$, because differentiating x makes it disappear.

Step 1. Choose the parts.

$u = x$ (differentiates to 1), $\frac{dv}{dx} = \sin x$, $v = -\cos x$:

$$\int x\sin x\,dx = -x\cos x - \int(-\cos x)\,dx = -x\cos x + \sin x + c.$$

Check: $\frac{d}{dx}(-x\cos x + \sin x) = -\cos x + x\sin x + \cos x = x\sin x$ ✓.

Step 2. Limits.

$$\Big[-x\cos x + \sin x\Big]_0^{\pi/2} = \left(-\tfrac{\pi}{2}\cdot 0 + 1\right) - (0 + 0) = 1.$$

Answer. 1

Common errors.

- $\int \sin x\,dx = \cos x$ (it is $-\cos x$).
- Choosing $u = \sin x$, which makes the new integral harder.

How the marks are earned.

M1 Parts with $u = x$: $-x\cos x + \int \cos x\,dx$
A1 $-x\cos x + \sin x$
M1 Substitutes limits
A1 1

∎

Question 11 *Vectors* 7 marks

Relative to a fixed origin O, the points A, B and C have position vectors

$$\mathbf{a} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}, \qquad \mathbf{b} = 4\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}, \qquad \mathbf{c} = 3\mathbf{j} + 5\mathbf{k}.$$

(a) *Find $|\overrightarrow{AB}|$.* *(2)*

(b) *Find the size of angle BAC, in degrees to one decimal place.* *(3)*

(c) *Find the exact area of triangle ABC.* *(2)*

Solution.

The idea.
Everything follows from two tools: vectors between points ("end minus start") and the scalar product $\mathbf{p}\cdot\mathbf{q} = |\mathbf{p}||\mathbf{q}|\cos\theta$. The area and the distance then come from basic triangle geometry.

Step 1. (a) The vector AB.

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix}, \qquad |\overrightarrow{AB}| = \sqrt{4 + 16 + 16} = \sqrt{36} = 6.$$

Step 2. (b) The vector AC.

$$\overrightarrow{AC} = \mathbf{c} - \mathbf{a} = \begin{pmatrix} 0 - 2 \\ 3 - 1 \\ 5 + 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 6 \end{pmatrix}, \qquad |\overrightarrow{AC}| = \sqrt{4 + 4 + 36} = \sqrt{44} = 2\sqrt{11}.$$

↪ *Both vectors must start at A, the vertex of the angle. Using $\overrightarrow{BA}$ by mistake would give the supplementary angle.*

Step 3. (b) Scalar product.

$$\overrightarrow{AB}\cdot\overrightarrow{AC} = (2)(-2) + (4)(2) + (4)(6) = -4 + 8 + 24 = 28.$$

$$\cos\theta = \frac{28}{6 \times 2\sqrt{11}} = \frac{7}{3\sqrt{11}} = 0.70352\ldots \quad \Longrightarrow \quad \theta = 45.3^\circ.$$

Step 4. (c) Area.
$\sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - \frac{49}{99}} = \sqrt{\frac{50}{99}}$. So

$$\text{Area} = \tfrac{1}{2} \times 6 \times 2\sqrt{11} \times \sqrt{\tfrac{50}{99}} = 6\sqrt{11} \cdot \frac{5\sqrt{2}}{3\sqrt{11}} = 10\sqrt{2} \approx 14.1.$$

Check with the vector product: $\overrightarrow{AB} \times \overrightarrow{AC} = (16, -20, 12)$, with length $\sqrt{800} = 20\sqrt{2}$; half is $10\sqrt{2}$ ✓.

Answer. (a) 6 (b) 45.3° (c) $10\sqrt{2}$

How the marks are earned.

M1 (a) $\overrightarrow{AB} = 2\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$

A1 6

M1 (b) $\overrightarrow{AC} = -2\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$

M1 $\cos\theta = \dfrac{28}{6\sqrt{44}}$

A1 45.3°

M1 (c) $\frac{1}{2} \times 6 \times \sqrt{44}\sin\theta$ or $\frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$

A1 $10\sqrt{2}$

∎

Question 12 *Vectors: parallelograms and distances* 4 marks

Relative to a fixed origin O, the points A, B and C have position vectors

$$\mathbf{a} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}, \qquad \mathbf{b} = 4\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}, \qquad \mathbf{c} = 3\mathbf{j} + 5\mathbf{k}.$$

You are given that $|\overrightarrow{AB}| = 6$ and that the area of triangle ABC is $10\sqrt{2}$.

(a) *Find the position vector of the point D such that $ABCD$ is a parallelogram.* *(2)*

(b) *Find the exact shortest distance from C to the line through A and B.* *(2)*

Solution.

The idea.

*Opposite sides of a parallelogram are equal vectors, so $\overrightarrow{AD} = \overrightarrow{BC}$. The shortest distance from a point to a line is a perpendicular **height**, which the area of the triangle gives directly.*

Step 1. (a) The parallelogram.

In parallelogram $ABCD$ (vertices in order), $\overrightarrow{AD} = \overrightarrow{BC}$:

$$\mathbf{d} = \mathbf{a} + (\mathbf{c} - \mathbf{b}) = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -4 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}.$$

Check: the diagonals bisect each other, so $\frac{\mathbf{a}+\mathbf{c}}{2} = \frac{\mathbf{b}+\mathbf{d}}{2}$: $(1, 2, 2)$ both ✓.

Step 2. (b) Distance as a height.

The shortest distance from C to line AB is the perpendicular height h of triangle ABC on base AB. From $\text{Area} = \frac{1}{2} \times AB \times h$:

$$h = \frac{2 \times 10\sqrt{2}}{6} = \frac{10\sqrt{2}}{3} \approx 4.71.$$

Answer. (a) $-2\mathbf{i} - \mathbf{j} + \mathbf{k}$ (b) $\dfrac{10\sqrt{2}}{3}$

Common errors.

- $\mathbf{d} = \mathbf{a} + \mathbf{b} - \mathbf{c}$ (wrong vertex order).
- Using position vectors $\mathbf{a} \cdot \mathbf{b}$ instead of the direction vectors for the angle.

How the marks are earned.

M1 (a) $\mathbf{d} = \mathbf{a} + \mathbf{c} - \mathbf{b}$

A1 $-2\mathbf{i} - \mathbf{j} + \mathbf{k}$

M1 (b) Height $= \dfrac{2 \times \text{area}}{|AB|}$

A1 $\dfrac{10\sqrt{2}}{3}$

∎

Question 13 *Differential equations* 10 marks

Water drains from a tank. The depth of the water, h metres, t minutes after the water starts to drain, satisfies

$$\frac{dh}{dt} = -k\sqrt{h},$$

where k is a positive constant. Initially the depth is 16 m, and after 10 minutes it is 9 m.

(a) *Show that $2\sqrt{h} = 8 - 0.2t$.* ***(5)***

(b) *Find the time taken for the tank to empty.* ***(2)***

(c) *Find the time taken for the depth to fall from 16 m to 4 m.* ***(2)***

(d) *Give one criticism of the model.* ***(1)***

Solution.

The idea.

Separate the variables (all h on one side, all t on the other), integrate both sides, then use the two conditions: one fixes the constant of integration, the other fixes k.

Step 1. (a) Separate the variables.

$$\frac{1}{\sqrt{h}}\,dh = -k\,dt \quad\Longrightarrow\quad \int h^{-\frac{1}{2}}\,dh = \int -k\,dt.$$

Step 2. (a) Integrate.

$$\frac{h^{\frac{1}{2}}}{\frac{1}{2}} = -kt + c \quad\Longrightarrow\quad 2\sqrt{h} = -kt + c.$$

↪ *One constant is enough; constants from both sides combine.*

Step 3. (a) Use the initial depth.

$t = 0,\ h = 16$: $2\sqrt{16} = c$, so $c = 8$.

Step 4. (a) Use the second reading.

$t = 10,\ h = 9$: $2\sqrt{9} = 8 - 10k \Rightarrow 6 = 8 - 10k \Rightarrow k = 0.2$.

$$2\sqrt{h} = 8 - 0.2t. \quad \blacksquare$$

Step 5. (b) Empty means h equals 0.

$$0 = 8 - 0.2t \quad\Longrightarrow\quad t = 40 \text{ minutes}.$$

Step 6. (c) Depth 4 m.

$2\sqrt{4} = 4 = 8 - 0.2t \Rightarrow t = 20$ minutes.

↪ *The first 12 m take 20 minutes, the last 4 m take another 20. The tank drains more slowly as it empties, because the rate is proportional to* $\sqrt{h}$.

Step 7. (d) A criticism.

For example: the model assumes the outflow depends only on $\sqrt{h}$ (Torricelli's law) and ignores the tank's shape and friction; or near $h = 0$ real flow may stop before the tank is exactly empty.

Answer. (b) 40 minutes (c) 20 minutes

Common errors.

- Integrating $h^{-\frac{1}{2}}$ to $\frac{h^{-\frac{3}{2}}}{-\frac{3}{2}}$.
- Finding k before c, with the wrong constant.

How the marks are earned.

M1 **(a)** Separates: $\int h^{-\frac{1}{2}}\,dh = \int -k\,dt$
A1 $2\sqrt{h} = -kt + c$
M1 $t = 0,\ h = 16 \Rightarrow c = 8$
M1 $t = 10,\ h = 9 \Rightarrow 6 = 8 - 10k$
A1* $k = 0.2$, giving $2\sqrt{h} = 8 - 0.2t$
M1 **(b)** $h = 0 \Rightarrow 8 - 0.2t = 0$
A1 40 minutes
M1 **(c)** $4 = 8 - 0.2t$
A1 20 minutes
B1 **(d)** Any sensible criticism *(E.g. the rate may not follow* $\sqrt{h}$ *exactly near the bottom; the shape of the tank isn't considered.)*

∎

Question 14 *Curve sketching and area* 7 marks

The curve C has equation

$$y = (x^2 - 3)e^{-x}.$$

(a) *Show that* $\dfrac{dy}{dx} = -(x-3)(x+1)e^{-x}$. *(3)*

(b) *Find the exact coordinates of the stationary points of C, and determine their nature.* *(4)*

Solution.

The idea.

Product rule for the gradient; stationary points where it vanishes; integration by parts (twice) for the area; and for (d), a careful picture of the whole curve, including what happens far to the left and right. The video shows the line $y = k$ sweeping over the curve.

Step 1. (a) Product rule.

$u = x^2 - 3$, $u' = 2x$; $v = e^{-x}$, $v' = -e^{-x}$:

$$\frac{dy}{dx} = 2xe^{-x} - (x^2 - 3)e^{-x} = e^{-x}(-x^2 + 2x + 3).$$

Step 2. (a) Factorise.

$$-x^2 + 2x + 3 = -(x^2 - 2x - 3) = -(x-3)(x+1),$$

so $\dfrac{dy}{dx} = -(x-3)(x+1)e^{-x}$. ■

Step 3. (b) Where the gradient is zero.

$e^{-x} > 0$ for all x, so $\frac{dy}{dx} = 0$ only when $x = 3$ or $x = -1$.

$$x = 3:\ y = 6e^{-3}, \qquad x = -1:\ y = (1-3)e^{1} = -2e.$$

Step 4. (b) Nature from the sign.

The sign of $\frac{dy}{dx}$ is the sign of $-(x-3)(x+1)$, a downward parabola in x:

x	$x < -1$	$-1 < x < 3$	$x > 3$
$\frac{dy}{dx}$	$-$	$+$	$-$

So $(-1, -2e)$ is a **minimum** (falling then rising) and $(3, 6e^{-3})$ is a **maximum**.

Answer. (b) minimum $(-1, -2e)$, maximum $(3, 6e^{-3})$

How the marks are earned.

M1 (a) Product rule: $2xe^{-x} - (x^2 - 3)e^{-x}$

A1 $e^{-x}(-x^2 + 2x + 3)$

A1* $-(x - 3)(x + 1)e^{-x}$

M1 (b) $x = 3,\ x = -1$

A1 $(3, 6e^{-3})$ and $(-1, -2e)$

M1 Sign of $\dfrac{dy}{dx}$ either side, or second derivative

A1 $(-1, -2e)$ minimum, $(3, 6e^{-3})$ maximum

∎

Question 15 *Area and intersections* 11 marks

> *The curve C has equation $y = (x^2 - 3)e^{-x}$. It has a minimum point at $(-1, -2e)$ and a maximum point at $(3, 6e^{-3})$.*
>
> ***(a)*** *Find the exact area of the finite region bounded by C and the x-axis.* ***(8)***
>
> ***(b)*** *Find the set of values of k for which the line $y = k$ meets C at exactly three points.* ***(3)***

A video explanation of this question is at lemmamathematics.com/papers/pure-3#q15.

Solution.

The idea.

*Integrate by parts twice for the area (the region is below the axis, so take its size). For the intersections, picture the **whole** curve, including what happens far to the left and right. The video shows the line $y = k$ sweeping over the curve.*

Step 1. (a) Where is the region.

$y = 0$ when $x^2 = 3$, so at $x = \pm\sqrt{3}$. Between these, $x^2 - 3 < 0$, so the curve is **below** the axis: the integral will be negative and I'll take its size.

Step 2. (a) Integrate x squared e to the minus x.

By parts twice. First, $u = x^2$, $v' = e^{-x}$, $v = -e^{-x}$:

$$\int x^2 e^{-x}\,dx = -x^2 e^{-x} + \int 2xe^{-x}\,dx.$$

Again, $u = 2x$, $v = -e^{-x}$:

$$\int 2xe^{-x}\,dx = -2xe^{-x} + \int 2e^{-x}\,dx = -2xe^{-x} - 2e^{-x}.$$

So $\int x^2 e^{-x}\,dx = -e^{-x}(x^2 + 2x + 2)$.

Step 3. (a) The full antiderivative.

$\int -3e^{-x}\,dx = 3e^{-x}$, so

$$F(x) = -e^{-x}(x^2 + 2x + 2) + 3e^{-x} = -e^{-x}(x^2 + 2x - 1).$$

Check: $F'(x) = e^{-x}(x^2 + 2x - 1) - e^{-x}(2x + 2) = e^{-x}(x^2 - 3)$ ✓.

Step 4. (a) Substitute the limits.

$$F(\sqrt{3}) = -e^{-\sqrt{3}}(3 + 2\sqrt{3} - 1) = -(2 + 2\sqrt{3})e^{-\sqrt{3}},$$

$$F(-\sqrt{3}) = -e^{\sqrt{3}}(3 - 2\sqrt{3} - 1) = -(2 - 2\sqrt{3})e^{\sqrt{3}} = (2\sqrt{3} - 2)e^{\sqrt{3}}.$$

$$\int_{-\sqrt{3}}^{\sqrt{3}} y\,dx = F(\sqrt{3}) - F(-\sqrt{3}) = -(2+2\sqrt{3})e^{-\sqrt{3}} - (2\sqrt{3}-2)e^{\sqrt{3}} \approx -9.24.$$

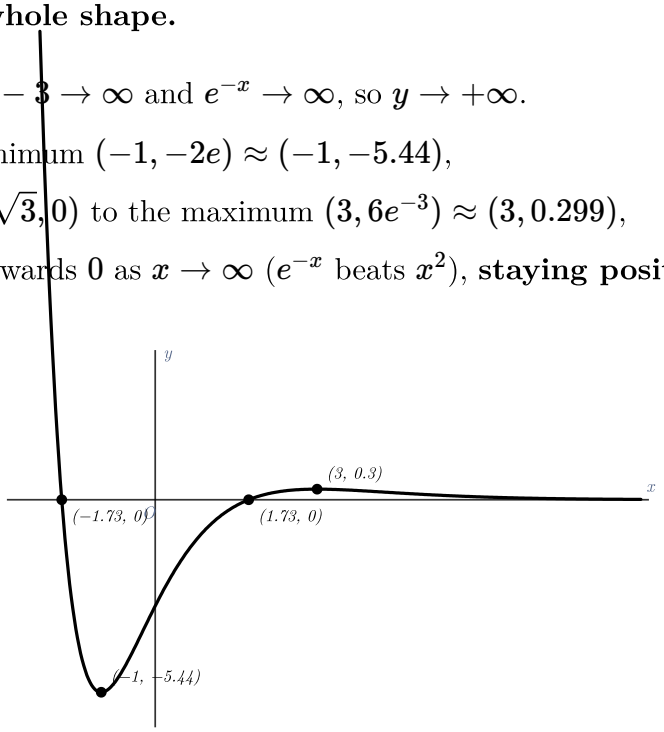
Step 5. (a) The area.

The region is below the axis, so

$$\text{Area} = (2\sqrt{3}-2)e^{\sqrt{3}} + (2\sqrt{3}+2)e^{-\sqrt{3}} \approx 8.275 + 0.967 = 9.24.$$

Step 6. (b) The whole shape.

- As $x \to -\infty$: $x^2 - 3 \to \infty$ and $e^{-x} \to \infty$, so $y \to +\infty$.
- It falls to the minimum $(-1, -2e) \approx (-1, -5.44)$,
- rises through $(\pm\sqrt{3}, 0)$ to the maximum $(3, 6e^{-3}) \approx (3, 0.299)$,
- then decreases towards 0 as $x \to \infty$ (e^{-x} beats x^2), **staying positive**.



Step 7. (b) Count the intersections.

Imagine sliding the horizontal line $y = k$ (see the video):

- $k > 6e^{-3}$: one point (far left only).
- $0 < k < 6e^{-3}$: one on the far left, and two around the hump on the right. **Three** ✓
- $k = 6e^{-3}$: two (touching at the maximum). $k = 0$: two ($\pm\sqrt{3}$; the right tail never reaches 0).
- $-2e < k < 0$: two. $k = -2e$: one.

$$0 < k < 6e^{-3}.$$

Answer. (a) $(2\sqrt{3}-2)e^{\sqrt{3}} + (2\sqrt{3}+2)e^{-\sqrt{3}} \approx 9.24$ (b) $0 < k < 6e^{-3}$

Common errors.

- Missing that $y \to +\infty$ on the left, which gives the extra intersection in (b).

- Including $k = 0$: the right-hand tail approaches 0 but never reaches it.
- Giving the area as negative.

Examiner's note.

For "how many intersections" questions, always sketch first. The algebra alone almost never shows the behaviour at infinity.

How the marks are earned.

B1 **(a)** Roots $x = \pm\sqrt{3}$

M1 Parts on x^2e^{-x} (applied twice), or a valid antiderivative method

A1 $\int x^2e^{-x}\,dx = -e^{-x}(x^2 + 2x + 2)$

A1 Antiderivative $F(x) = -e^{-x}(x^2 + 2x - 1)$

M1 Substitutes $\pm\sqrt{3}$

A1 $F(\sqrt{3}) = -(2 + 2\sqrt{3})e^{-\sqrt{3}}$ and $F(-\sqrt{3}) = (2\sqrt{3} - 2)e^{\sqrt{3}}$

M1 Takes the modulus (region below the axis)

A1 $(2\sqrt{3} - 2)e^{\sqrt{3}} + (2\sqrt{3} + 2)e^{-\sqrt{3}}$ *(≈ 9.24)*

M1 **(b)** Uses the behaviour as $x \to \pm\infty$ *($y \to \infty$ as $x \to -\infty$; $y \to 0^+$ as $x \to \infty$.)*

A1 Upper bound $6e^{-3}$

A1 $0 < k < 6e^{-3}$

∎