LEMMA PRACTICE PAPER · MARK SCHEME

Level 4

Pure Mathematics

Paper 4 · Beyond the exam

GENERAL MARKING GUIDANCE

- All candidates must receive the same treatment: mark positively, and give credit for what is shown rather than penalising what is missing.
- **M** marks: method marks, for knowing and applying a correct method to the problem, even if there are small slips.
- **A** marks: accuracy marks, which can only be awarded if the relevant M mark has been earned.
- **B** marks: unconditional accuracy marks, independent of method.
- **dM**: a method mark that depends on the previous M mark. **ft**: follow through from an earlier error. *****: the answer is given in the question, so full working must be seen with no errors. **cao**: correct answer only.
- Correct alternative methods earn full credit. The model solutions show several of them.

QUESTION	SCHEME	MARKS	NOTES
1	Shows $x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4} > 0$	**B1**	So the expression is defined for all x and inequalities can be cleared.
	Sets $y = \dfrac{x^2 + x + 1}{x^2 - x + 1}$ and rearranges to a quadratic in x	**M1**	
	$(y-1)x^2 - (y+1)x + (y-1) = 0$	**A1**	
	Real x requires discriminant ≥ 0 (for $y \neq 1$)	**M1**	
	$(y+1)^2 - 4(y-1)^2 \geq 0$	**A1**	
	Factorises as a difference of two squares	**M1**	
	$(3-y)(3y-1) \geq 0$	**A1**	
	$\frac{1}{3} \leq y \leq 3$, with $y = 1$ (at $x = 0$) consistent	**A1**	
	$y = 3$ at $x = 1$; $y = \frac{1}{3}$ at $x = -1$	**B1**	
	Total for Question 1	**(9)**	

QUESTION	SCHEME	MARKS	NOTES
2 (a)	Common denominator $r(r+1)(r+2)$	**M1**	
	Numerator $(r+2) - r = 2$	**A1**	
	Completes with the factor $\frac{1}{2}$	**A1***	
(b)	Writes out the first and last terms of the sum	**M1**	
	Shows the cancellation (telescoping)	**M1**	

QUESTION	SCHEME	MARKS	NOTES
	$\frac{1}{2}\left(\frac{1}{2}-\frac{1}{(n+1)(n+2)}\right)$	**A1**	
	$=\frac{1}{4}-\frac{1}{2(n+1)(n+2)}$	**A1***	
(c)	$\frac{1}{4}$	**B1**	
	Total for Question 2	**(8)**	

QUESTION	SCHEME	MARKS	NOTES
3	$\dfrac{1}{2(n+1)(n+2)} < 0.0001$	**M1**	
	$(n+1)(n+2) > 5000$	**M1**	
	$n = 70$ (checks $70 \times 71 = 4970$, $71 \times 72 = 5112$)	**A1**	
	Total for Question 3	**(3)**	

QUESTION	SCHEME	MARKS	NOTES
4	$\tan(2\theta+\theta) = \dfrac{\tan 2\theta + \tan\theta}{1-\tan 2\theta\tan\theta}$	**M1**	
	Substitutes $\tan 2\theta = \dfrac{2t}{1-t^2}$	**M1**	
	Multiplies top and bottom by $1-t^2$	**M1**	
	Completes to the given result	**A1***	
	Total for Question 4	**(4)**	

QUESTION	SCHEME	MARKS	NOTES
5 (a)	$\tan\dfrac{3\pi}{12} = \tan\dfrac{\pi}{4} = 1$	**M1**	
	$3t - t^3 = 1 - 3t^2 \Rightarrow t^3 - 3t^2 - 3t + 1 = 0$	**A1***	
(b)	Spots $t = -1$ as a root	**M1**	
	$(t+1)(t^2-4t+1)$	**A1**	
	$t = 2 \pm \sqrt{3}$	**M1**	
	$\tan\dfrac{\pi}{12} = 2-\sqrt{3}$, since $0 < \tan\frac{\pi}{12} < \tan\frac{\pi}{4} = 1$	**A1**	
	Total for Question 5	**(6)**	

QUESTION	SCHEME	MARKS	NOTES
6 (a)	$dx = -du$ and limits $\frac{\pi}{2} \to 0$, $0 \to \frac{\pi}{2}$	**M1**	
	$\sin\left(\frac{\pi}{2}-u\right) = \cos u$ and $\cos\left(\frac{\pi}{2}-u\right) = \sin u$	**M1**	
	Obtains J (dummy variable noted)	**A1***	
(b)	$I + J = \int_0^{\pi/2} 1\,dx$	**M1**	
	$= \frac{\pi}{2}$	**A1**	

QUESTION	SCHEME	MARKS	NOTES
	$I = \frac{\pi}{4}$	**A1**	
(c)	Writes the integrand as $\dfrac{\cos^k x}{\cos^k x + \sin^k x}$ and applies the same argument	**M1**	
	$\frac{\pi}{4}$	**A1**	
	Total for Question 6	**(8)**	

QUESTION	SCHEME	MARKS	NOTES
7	Uses $x = \pi - u$	**M1**	
	$K = \int_0^{\pi} \dfrac{(\pi - u)\sin u}{1 + \cos^2 u}\, du$	**A1**	
	$2K = \pi \int_0^{\pi} \dfrac{\sin x}{1 + \cos^2 x}\, dx$	**M1**	
	$u = \cos x$ gives $\int_{-1}^{1} \dfrac{du}{1 + u^2} = \frac{\pi}{2}$	**M1**	
	$K = \dfrac{\pi^2}{4}$	**A1**	
	Total for Question 7	**(5)**	

QUESTION	SCHEME	MARKS	NOTES
8	Separates: $\int \dfrac{5000}{P(500 - P)}\, dP = \int dt$	**M1**	
	Partial fractions $\dfrac{A}{P} + \dfrac{B}{500 - P}$	**M1**	
	$A = B = 10$	**A1**	
	$10 \ln P - 10 \ln(500 - P) = t + c$	**M1**	
	$\ln \dfrac{P}{500 - P} = 0.1t + c'$	**A1**	
	Uses $t = 0,\ P = 50$: $c' = \ln \frac{1}{9}$	**M1**	
	Rearranges to the given form	**A1***	
	Total for Question 8	**(7)**	

QUESTION	SCHEME	MARKS	NOTES
9 (a)	$1 + 9e^{-0.1t} = 2$	**M1**	
	$t = 10 \ln 9 \approx 22.0$ years	**A1**	
(b)	Tends to 500	**B1**	
(c)	$\dfrac{dP}{dt} = 0.1P - \dfrac{P^2}{5000}$, a quadratic in P	**M1**	
	Maximum at $P = 250$ (vertex or derivative)	**A1***	
	12.5 fish per year	**A1**	

QUESTION	SCHEME	MARKS	NOTES
	Total for Question 9	**(6)**	

QUESTION	SCHEME	MARKS	NOTES
10	Midpoint of BC: $\frac{1}{2}(\mathbf{b}+\mathbf{c})$	**M1**	
	Point $\frac{2}{3}$ of the way from A: $\mathbf{a}+\frac{2}{3}\left(\frac{1}{2}(\mathbf{b}+\mathbf{c})-\mathbf{a}\right)$	**M1**	
	$=\frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c})$	**A1**	
	Same calculation (or symmetry) for the medians from B and C	**M1**	
	All give $\frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c})$	**A1**	
	Concludes: a common point, dividing each median $\mathbf{2:1}$	**A1**	
	Total for Question 10	**(6)**	

QUESTION	SCHEME	MARKS	NOTES
11 (a)	Equates components	**M1**	
	Solves two equations	**M1**	
	$\lambda=\frac{4}{3},\ \mu=-\frac{1}{3}$	**A1**	
	Checks the third equation	**M1**	
	$\left(\frac{7}{3},\frac{2}{3},\frac{17}{3}\right)$	**A1**	
(b)	Scalar product of the direction vectors: $\mathbf{3}$	**M1**	
	$\cos\theta=\dfrac{3}{\sqrt{6}\sqrt{6}}=\frac{1}{2}$	**M1**	
	60°	**A1**	
	Total for Question 11	**(8)**	

QUESTION	SCHEME	MARKS	NOTES
12 (a)	Considers $f(x)=e^x-1-x$	**M1**	
	$f'(x)=e^x-1=0 \iff x=0$	**M1**	
	Minimum (by $f''>0$ or sign change), $f(0)=0$	**A1**	
	Concludes $f(x)\geqslant 0$, equality only at $x=0$	**A1**	
(b)	$x=\frac{1}{n}$: $e^{1/n}>1+\frac{1}{n}$	**M1**	
	Raises to the power n (both sides positive)	**A1***	
	Total for Question 12	**(6)**	

QUESTION	SCHEME	MARKS	NOTES
13 (a)	$x=-\dfrac{1}{n+1}$	**M1**	

QUESTION	SCHEME	MARKS	NOTES
	$e^{-1/(n+1)} > \dfrac{n}{n+1}$	**A1**	
	Takes reciprocals (reversing) and raises to the power $n+1$	**M1**	
	$e < \left(\dfrac{n+1}{n}\right)^{n+1}$	**A1***	
(b)	Uses $n = 10$	**M1**	
	$1.1^{10} = 2.5937\ldots > 2.59$	**A1**	
	$1.1^{11} = 2.8531\ldots < 2.86$	**A1**	
	Total for Question 13	**(7)**	

QUESTION	SCHEME	MARKS	NOTES
14	$V_c = 2\pi y(R^2 - y^2)$ with half-height y	**M1**	
	$\dfrac{dV_c}{dy} = 2\pi(R^2 - 3y^2) = 0$	**M1**	
	$y = \dfrac{R}{\sqrt{3}}$	**A1**	
	$V_c = \dfrac{4\sqrt{3}\,\pi R^3}{9}$	**A1**	
	Total for Question 14	**(4)**	

QUESTION	SCHEME	MARKS	NOTES
15 (a)	Cross-section: base radius r with $r^2 + (h-R)^2 = R^2$	**M1**	
	$r^2 = 2Rh - h^2$	**A1**	
	$V = \frac{1}{3}\pi r^2 h$	**M1**	
	$V = \frac{\pi}{3}(2Rh^2 - h^3)$	**A1***	
(b)	$\dfrac{dV}{dh} = \dfrac{\pi}{3}(4Rh - 3h^2)$	**M1**	
	Correct derivative	**A1**	
	Sets $= 0$ and solves	**M1**	
	$h = \dfrac{4R}{3}$	**A1**	
	$\dfrac{d^2V}{dh^2} = \dfrac{\pi}{3}(4R - 6h) < 0$ at $h = \frac{4R}{3}$	**M1**	
	Concludes maximum	**A1**	
	$V_{\max} = \dfrac{32\pi R^3}{81}$	**A1**	
(c)	$\dfrac{32\pi R^3/81}{4\pi R^3/3}$	**M1**	
	$= \dfrac{8}{27}$	**A1***	

QUESTION	SCHEME	MARKS	NOTES
	Total for Question 15	**(13)**	

TOTAL FOR PAPER: 100 MARKS