Name _____ Date _____ Score _____ / 100

LEMMA PRACTICE PAPER

Level 4

Pure Mathematics

Paper 4 · Beyond the exam

Paper reference **LP4/P** Time **2 hours 30 minutes**

TOTAL MARKS	100

You must have: pen, pencil, ruler, graph paper and a calculator.

INSTRUCTIONS

- Use black ink or ball-point pen; pencil may be used for diagrams.
- Answer **all** questions, in the spaces provided. If you need more space, continue on extra paper and label it clearly.
- Show all your working. An answer with no supporting method may get no marks.
- Give answers to 3 significant figures unless a question says otherwise, and give exact answers when asked.

INFORMATION

- There are **15** questions, and the paper is out of **100** marks.
- The marks for each part of a question are shown in brackets, for example **(3)**. Use them to judge how much to write.
- The questions are in roughly increasing order of difficulty.
- Difficulty: **Beyond the exam**. Harder than any real A level paper. Unfamiliar set-ups, proofs you have to plan yourself and long questions that build to a result. Good preparation for university interviews and entrance tests.

ADVICE

- Read each question carefully before you start.
- Keep an eye on the time: roughly one minute per mark.
- Check your answers if you have time at the end.
- The mark scheme, very detailed model solutions and video explanations are free at **lemmamathematics.com/papers/pure-4**.

1 Prove that, for all real values of x, **(9)**

$$\frac{1}{3} \leq \frac{x^2 + x + 1}{x^2 - x + 1} \leq 3,$$

and find the values of x for which each bound is attained.

(Total for Question 1 is 9 marks)

2 **(a)** Show that **(3)**

$$\frac{1}{r(r+1)(r+2)} \equiv \frac{1}{2}\left(\frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)}\right).$$

(b) Hence show that **(4)**

$$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}.$$

(c) State the value of $\displaystyle\sum_{r=1}^{\infty} \frac{1}{r(r+1)(r+2)}$. **(1)**

(Total for Question 2 is 8 marks)

3 You are given that **(3)**

$$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}.$$

Find the least value of n for which $\displaystyle\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} > 0.2499$.

(Total for Question 3 is 3 marks)

4 Prove that **(4)**

$$\tan 3\theta \equiv \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}.$$

(Total for Question 4 is 4 marks)

5 You are given that

$$\tan 3\theta \equiv \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}.$$

(a) Hence show that $t = \tan\dfrac{\pi}{12}$ satisfies **(2)**

$$t^3 - 3t^2 - 3t + 1 = 0.$$

(b) Hence find the exact value of $\tan \frac{\pi}{12}$, justifying your choice of root. **(4)**

(Total for Question 5 is 6 marks)

6 Let

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\, dx \qquad \text{and} \qquad J = \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + \cos x}\, dx.$$

(a) By using the substitution $x = \frac{\pi}{2} - u$, show that $I = J$. **(3)**

(b) Hence find the exact value of I. **(3)**

(c) Deduce, for any positive integer k, the value of $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1+\tan^k x}\,dx$. **(2)**

(Total for Question 6 is 8 marks)

7 By using a suitable substitution, find the exact value of **(5)**

$$\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} \, dx.$$

(Total for Question 7 is 5 marks)

8 The population, P, of a species of fish in a lake, t years after it was introduced, is modelled by **(7)**

$$\frac{dP}{dt} = 0.1P\left(1 - \frac{P}{500}\right).$$

Initially there are 50 fish.

Solve the differential equation to show that

$$P = \frac{500}{1 + 9e^{-0.1t}}.$$

(Total for Question 8 is 7 marks)

9 The number of fish, P, in a lake t years after they were introduced satisfies

$$\frac{dP}{dt} = 0.1P\left(1 - \frac{P}{500}\right), \qquad P = \frac{500}{1 + 9e^{-0.1t}}.$$

(a) Find the time taken for the population to reach 250. **(2)**

(b) State what happens to the population in the long term. **(1)**

(c) Show that the population is growing fastest when $P = 250$, and find this maximum rate of growth. **(3)**

(Total for Question 9 is 6 marks)

10 The points A, B and C have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. A median of triangle ABC joins a vertex to the midpoint of the opposite side. Prove, using vectors, that the three medians of triangle ABC meet at a single point, and that this point divides each median in the ratio $\mathbf{2 : 1}$. **(6)**

(Total for Question 10 is 6 marks)

11 The lines l_1 and l_2 have equations

$$l_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \qquad l_2 : \mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$

(a) Show that l_1 and l_2 intersect, and find the coordinates of their point of intersection. **(5)**

(b) Find the acute angle between l_1 and l_2. **(3)**

(Total for Question 11 is 8 marks)

12 **(a)** Prove that $e^x \geq 1 + x$ for all real x, with equality only when $x = 0$. **(4)**

(b) By choosing a suitable value of x, deduce that, for every positive integer n, **(2)**

$$\left(1 + \frac{1}{n}\right)^n < e.$$

(Total for Question 12 is 6 marks)

13 You may assume that $e^x \geq 1 + x$ for all real x, with equality only when $x = 0$, and that $\left(1 + \frac{1}{n}\right)^n < e$ for every positive integer n.

(a) By choosing another suitable value of x, show that, for every positive integer n, **(4)**

$$e < \left(1 + \frac{1}{n}\right)^{n+1}.$$

(b) Hence show that $2.59 < e < 2.86$. **(3)**

(Total for Question 13 is 7 marks)

14 A right circular cylinder is inscribed in a sphere of radius R: both circular edges of the cylinder lie on the sphere. Find the maximum possible volume of the cylinder, justifying that it is a maximum. **(4)**

(Total for Question 14 is 4 marks)

15 A right circular cone is inscribed in a sphere of radius R: its vertex and the whole circumference of its base lie on the sphere. The height of the cone is h, where $0 < h < 2R$.

(a) Show that the volume of the cone is **(4)**

$$V = \frac{\pi}{3}\left(2Rh^2 - h^3\right).$$

(b) Find, in terms of R, the maximum possible volume of the cone, justifying that it is a maximum. **(7)**

(c) Show that the largest cone fills exactly $\frac{8}{27}$ of the sphere. **(2)**

(Total for Question 15 is 13 marks)

TOTAL FOR PAPER IS 100 MARKS

END OF PAPER