Pure Mathematics

Paper 4 · Beyond the exam: model solutions

LP4/P · Level 4

How to use these solutions. Try each question under timed conditions first, then mark it with the mark scheme. Read the full solution for every question, even the ones you got right: they often show a faster method, a check, or a common error to avoid. Each solution begins with *the idea*, then works through numbered steps; remarks marked ↪ explain *why*. Each ends with the boxed answer and how the marks are earned.

Question 1 *Ranges of functions* 9 marks

Prove that, for all real values of x,

$$\frac{1}{3} \leq \frac{x^2+x+1}{x^2-x+1} \leq 3,$$

and find the values of x for which each bound is attained.

Solution.

The idea.

A neat trick: call the value of the expression y, and ask "for which y is there a real x giving this value?" Clearing the fraction gives a quadratic in x, and "has a real solution" means "discriminant ≥ 0". That produces an inequality in y, which is exactly the range.

Step 1. The denominator is never zero.

Complete the square:

$$x^2-x+1 = \left(x-\tfrac{1}{2}\right)^2 + \tfrac{3}{4} \geq \tfrac{3}{4} > 0.$$

So the fraction is defined for every real x, and multiplying through by the denominator never flips an inequality.

Step 2. Name the value.

Let

$$y = \frac{x^2+x+1}{x^2-x+1}.$$

A number y is in the range exactly when this equation has a real solution x.

Step 3. Make a quadratic in x.

$$y(x^2-x+1) = x^2+x+1 \quad\Longrightarrow\quad yx^2-yx+y-x^2-x-1 = 0,$$

$$(y-1)x^2-(y+1)x+(y-1)=0.$$

Step 4. The special case y equals 1.

If $y=1$ the x^2 terms cancel: $-2x=0$, so $x=0$. Indeed the expression equals $\frac{1}{1}=1$ at $x=0$. Fine, and 1 lies within $\left[\frac{1}{3},3\right]$.

Step 5. Otherwise use the discriminant.

For $y\neq 1$ it's a genuine quadratic, with a real root if and only if

$$(y+1)^2-4(y-1)(y-1)\geq 0 \quad\Longrightarrow\quad (y+1)^2-\big(2(y-1)\big)^2\geq 0.$$

Step 6. Difference of two squares.

$A^2-B^2=(A-B)(A+B)$ with $A=y+1$, $B=2y-2$:

$$\big(y+1-2y+2\big)\big(y+1+2y-2\big)\geq 0 \quad\Longrightarrow\quad (3-y)(3y-1)\geq 0.$$

↪ *Factorising like this is much cleaner than expanding everything into $-3y^2+10y-3\geq 0$, though that's equivalent.*

Step 7. Solve the inequality.

$(3-y)(3y-1)$ is a downward-opening quadratic in y with roots $\frac{1}{3}$ and 3, so it is ≥ 0 between them:

$$\frac{1}{3}\leq y\leq 3.$$

Every value of the expression lies in $\left[\frac{1}{3},3\right]$. ■

Step 8. When the bounds are attained.

A bound is attained when the discriminant is 0 (a repeated root), $x=\dfrac{y+1}{2(y-1)}$:

- $y=3$: $x=\dfrac{4}{4}=1$. Check: $\frac{1+1+1}{1-1+1}=3$ ✓
- $y=\frac{1}{3}$: $x=\dfrac{4/3}{-4/3}=-1$. Check: $\frac{1-1+1}{1+1+1}=\frac{1}{3}$ ✓

Answer. $\frac{1}{3}\leq\dfrac{x^2+x+1}{x^2-x+1}\leq 3$; the maximum 3 occurs at $x=1$ and the minimum $\frac{1}{3}$ at $x=-1$.

Another method.

By calculus. Differentiating gives $\frac{dy}{dx}=\frac{2(1-x^2)}{(x^2-x+1)^2}$, which is zero at $x=\pm 1$, and $y\to 1$ as $x\to\pm\infty$. So the extreme values are $y(1)=3$ and $y(-1)=\frac{1}{3}$. The discriminant method avoids the quotient rule and the limit argument.

Common errors.

- Forgetting to justify that the denominator is positive.
- Ignoring the case $y=1$, where the "quadratic" isn't one.

How the marks are earned.

B1 Shows $x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4} > 0$ *(So the expression is defined for all x and inequalities can be cleared.)*

M1 Sets $y = \dfrac{x^2 + x + 1}{x^2 - x + 1}$ and rearranges to a quadratic in x

A1 $(y-1)x^2 - (y+1)x + (y-1) = 0$

M1 Real x requires discriminant ≥ 0 (for $y \neq 1$)

A1 $(y+1)^2 - 4(y-1)^2 \geq 0$

M1 Factorises as a difference of two squares

A1 $(3-y)(3y-1) \geq 0$

A1 $\frac{1}{3} \leq y \leq 3$, with $y = 1$ (at $x = 0$) consistent

B1 $y = 3$ at $x = 1$; $y = \frac{1}{3}$ at $x = -1$

∎

Question 2 *Telescoping series* 8 marks

(a) *Show that* *(3)*

$$\frac{1}{r(r+1)(r+2)} \equiv \frac{1}{2}\left(\frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)}\right).$$

(b) *Hence show that* *(4)*

$$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}.$$

(c) *State the value of* $\displaystyle\sum_{r=1}^{\infty} \frac{1}{r(r+1)(r+2)}$. *(1)*

Solution.

The idea.

Writing each term as a ***difference*** *of consecutive values of one function makes the sum collapse: almost everything cancels, like a telescope folding up. Then the infinite sum is just a limit.*

Step 1. (a) Combine the right-hand side.

Over the common denominator $r(r+1)(r+2)$:

$$\frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)} = \frac{(r+2)-r}{r(r+1)(r+2)} = \frac{2}{r(r+1)(r+2)}.$$

Halving gives exactly the left-hand side. ■

Step 2. (b) Name the pattern.

Let $g(r) = \dfrac{1}{r(r+1)}$. Part (a) says the rth term is $\frac{1}{2}\big(g(r) - g(r+1)\big)$.

Step 3. (b) Write the sum out.

$$\sum_{r=1}^{n} \tfrac{1}{2}\big(g(r) - g(r+1)\big) = \tfrac{1}{2}\Big[\big(g(1) - g(2)\big) + \big(g(2) - g(3)\big) + \cdots + \big(g(n) - g(n+1)\big)\Big].$$

Every $g(2), g(3), \ldots, g(n)$ appears once with $+$ and once with $-$, so they cancel.

Step 4. (b) What survives.

Only the first and last terms are left:

$$\tfrac{1}{2}\big(g(1) - g(n+1)\big) = \tfrac{1}{2}\left(\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right) = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}. \quad ■$$

Check $n = 1$: the sum is $\frac{1}{6}$, and the formula gives $\frac{1}{4} - \frac{1}{12} = \frac{1}{6}$ ✓.

Step 5. (c) Let n go to infinity.

$\frac{1}{2(n+1)(n+2)} \to 0$, so the sum to infinity is $\dfrac{1}{4}$.

Answer. (c) $\frac{1}{4}$

Common errors.

- Cancelling terms you can't see: write out at least three at the start and two at the end.

How the marks are earned.

M1 **(a)** Common denominator $r(r+1)(r+2)$
A1 Numerator $(r+2)-r=2$
A1* Completes with the factor $\frac{1}{2}$
M1 **(b)** Writes out the first and last terms of the sum
M1 Shows the cancellation (telescoping)
A1 $\frac{1}{2}\left(\frac{1}{2}-\frac{1}{(n+1)(n+2)}\right)$
A1* $=\frac{1}{4}-\frac{1}{2(n+1)(n+2)}$
B1 **(c)** $\frac{1}{4}$

∎

Question 3 *Series and inequalities* 3 marks

You are given that

$$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}.$$

Find the least value of n *for which* $\displaystyle\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} > 0.2499$.

Solution.

The idea.

The sum is $\frac{1}{4}$ *minus a small positive "gap". Asking for the sum to exceed* 0.2499 *is asking for the gap to be less than* 0.0001*, which is an inequality in* n*.*

Step 1. Turn it into an inequality.

$$\frac{1}{4} - \frac{1}{2(n+1)(n+2)} > 0.2499 \quad\Longrightarrow\quad \frac{1}{2(n+1)(n+2)} < 0.0001 \quad\Longrightarrow\quad (n+1)(n+2) > 5000.$$

Step 2. Find the least n.

$\sqrt{5000} \approx 70.7$, so try values around there:

- $n = 69$: $70 \times 71 = 4970$, not enough.
- $n = 70$: $71 \times 72 = 5112 > 5000$ ✓.

The least value is $n = 70$.

Answer. $n = 70$

How the marks are earned.

M1 $\dfrac{1}{2(n+1)(n+2)} < 0.0001$
M1 $(n+1)(n+2) > 5000$
A1 $n = 70$ (checks $70 \times 71 = 4970$, $71 \times 72 = 5112$)

∎

Question 4 *Multiple angles* 4 marks

Prove that

$$\tan 3\theta \equiv \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}.$$

Solution.

The idea.

Build 3θ as $2\theta + \theta$ and use the addition formula. Then putting $\theta = \frac{\pi}{12}$ makes $\tan 3\theta$ a known value, which turns the identity into an equation for $\tan\frac{\pi}{12}$.

Step 1. Split 3θ.

Let $t = \tan\theta$. Using $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$ with $A = 2\theta$, $B = \theta$:

$$\tan 3\theta = \frac{\tan 2\theta + t}{1 - t\tan 2\theta}, \qquad \tan 2\theta = \frac{2t}{1 - t^2}.$$

Step 2. Substitute and clear fractions.

$$\tan 3\theta = \frac{\dfrac{2t}{1-t^2} + t}{1 - \dfrac{2t^2}{1-t^2}}.$$

Multiply numerator and denominator by $1 - t^2$:

$$\tan 3\theta = \frac{2t + t(1-t^2)}{(1-t^2) - 2t^2} = \frac{3t - t^3}{1 - 3t^2}. \quad \blacksquare$$

Answer. Proof as above.

How the marks are earned.

M1 $\tan(2\theta+\theta) = \dfrac{\tan 2\theta + \tan\theta}{1 - \tan 2\theta\tan\theta}$

M1 Substitutes $\tan 2\theta = \dfrac{2t}{1-t^2}$

M1 Multiplies top and bottom by $1 - t^2$

A1* Completes to the given result

∎

Question 5 *Exact values from cubics* 6 marks

You are given that

$$\tan 3\theta \equiv \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}.$$

(a) *Hence show that* $t = \tan\dfrac{\pi}{12}$ *satisfies* *(2)*

$$t^3 - 3t^2 - 3t + 1 = 0.$$

(b) *Hence find the exact value of* $\tan\dfrac{\pi}{12}$*, justifying your choice of root.* *(4)*

Solution.

The idea.

Choosing $\theta = \frac{\pi}{12}$ *makes* $3\theta = \frac{\pi}{4}$*, where the tangent is exactly 1. That turns the identity into a cubic equation for* $\tan\frac{\pi}{12}$*, which factorises.*

Step 1. (a) Choose θ equal to π over 12.

Then $3\theta = \frac{\pi}{4}$ and $\tan\frac{\pi}{4} = 1$:

$$1 = \frac{3t - t^3}{1 - 3t^2} \quad\Longrightarrow\quad 1 - 3t^2 = 3t - t^3 \quad\Longrightarrow\quad t^3 - 3t^2 - 3t + 1 = 0. \quad \blacksquare$$

Step 2. (b) Find an easy root.

Try $t = -1$: $-1 - 3 + 3 + 1 = 0$ ✓. So $(t + 1)$ is a factor. Comparing coefficients:

$$t^3 - 3t^2 - 3t + 1 = (t + 1)(t^2 - 4t + 1).$$

↪ *Why is* -1 *a root? The cubic is satisfied by* $\tan\theta$ *for* ***every*** θ *with* $\tan 3\theta = 1$*, including* $\theta = \frac{3\pi}{4}$*, where* $\tan\theta = -1$*.*

Step 3. (b) Solve the quadratic.

$$t = \frac{4 \pm \sqrt{16 - 4}}{2} = 2 \pm \sqrt{3}.$$

So the three roots are -1, $2 - \sqrt{3} \approx 0.268$ and $2 + \sqrt{3} \approx 3.732$.

Step 4. (b) Pick the right root.

$0 < \frac{\pi}{12} < \frac{\pi}{4}$, and $\tan$ is increasing there, so $0 < \tan\frac{\pi}{12} < 1$. Only one root qualifies:

$$\tan\frac{\pi}{12} = 2 - \sqrt{3}.$$

(The root $2 + \sqrt{3}$ is $\tan\frac{5\pi}{12}$, another angle whose triple has tangent 1.) **Check:** $\tan 15^\circ = 0.2679\ldots$ and $2 - \sqrt{3} = 0.2679\ldots$ ✓.

Answer. (b) $\tan\dfrac{\pi}{12} = 2 - \sqrt{3}$

Common errors.

- Choosing $2 + \sqrt{3}$ without justification.

How the marks are earned.

M1 (a) $\tan\dfrac{3\pi}{12}=\tan\dfrac{\pi}{4}=1$

A1* $3t-t^3=1-3t^2\Rightarrow t^3-3t^2-3t+1=0$

M1 (b) Spots $t=-1$ as a root

A1 $(t+1)(t^2-4t+1)$

M1 $t=2\pm\sqrt{3}$

A1 $\tan\dfrac{\pi}{12}=2-\sqrt{3}$, since $0<\tan\frac{\pi}{12}<\tan\frac{\pi}{4}=1$

∎

Question 6 *Symmetry in integrals* 8 marks

Let

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\, dx \qquad \text{and} \qquad J = \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + \cos x}\, dx.$$

(a) *By using the substitution* $x = \dfrac{\pi}{2} - u$*, show that* $I = J$*.* *(3)*

(b) *Hence find the exact value of* I*.* *(3)*

(c) *Deduce, for any positive integer* k*, the value of* $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1 + \tan^k x}\, dx$*.* *(2)*

Solution.

The idea.

Some integrals can't be done directly but have a hidden symmetry: the substitution $x \to (a+b) - x$ *maps the interval* $[a, b]$ *onto itself. Comparing the integral with its "mirror image" and adding gives something easy.*

Step 1. (a) Substitute.

$x = \frac{\pi}{2} - u$, so $dx = -du$; when $x = 0$, $u = \frac{\pi}{2}$, and when $x = \frac{\pi}{2}$, $u = 0$:

$$I = \int_{\pi/2}^{0} \frac{\sin\left(\frac{\pi}{2} - u\right)}{\sin\left(\frac{\pi}{2} - u\right) + \cos\left(\frac{\pi}{2} - u\right)}\, (-du).$$

Step 2. (a) Simplify.

Use $\sin\left(\frac{\pi}{2} - u\right) = \cos u$ and $\cos\left(\frac{\pi}{2} - u\right) = \sin u$, and the minus sign flips the limits back:

$$I = \int_0^{\pi/2} \frac{\cos u}{\cos u + \sin u}\, du = J. \quad \blacksquare$$

↪ *The name of the variable doesn't matter in a definite integral, so* $\int_0^{\pi/2} \frac{\cos u}{\cdots}\, du$ *is the same number as* J*.*

Step 3. (b) Add them.

$$I + J = \int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x}\, dx = \int_0^{\pi/2} 1\, dx = \frac{\pi}{2}.$$

Since $I = J$: $2I = \frac{\pi}{2}$, so

$$I = \frac{\pi}{4}.$$

Step 4. (c) Rewrite in sines and cosines.

$$\frac{1}{1 + \tan^k x} = \frac{\cos^k x}{\cos^k x + \sin^k x}.$$

The same substitution swaps $\sin^k$ and $\cos^k$, so this integral equals its partner with $\sin^k x$ on top. The two add to $\int_0^{\pi/2} 1\, dx = \frac{\pi}{2}$, so each is

$$\frac{\pi}{4},$$

for **every** k. A remarkable fact: the answer doesn't depend on k at all.

Answer. (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{4}$

How the marks are earned.

M1 **(a)** $dx = -du$ and limits $\frac{\pi}{2} \to 0$, $0 \to \frac{\pi}{2}$

M1 $\sin\left(\frac{\pi}{2} - u\right) = \cos u$ and $\cos\left(\frac{\pi}{2} - u\right) = \sin u$

A1* Obtains J (dummy variable noted)

M1 **(b)** $I + J = \int_0^{\pi/2} 1 \, dx$

A1 $= \frac{\pi}{2}$

A1 $I = \frac{\pi}{4}$

M1 **(c)** Writes the integrand as $\dfrac{\cos^k x}{\cos^k x + \sin^k x}$ and applies the same argument

A1 $\frac{\pi}{4}$

∎

Question 7 *Symmetry in integrals II* 5 marks

By using a suitable substitution, find the exact value of

$$\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\, dx.$$

Solution.

The idea.

Replace x by $\pi - x$: the interval $[0,\pi]$ maps to itself, $\sin x$ and $\cos^2 x$ are unchanged, and adding the two versions of the integral removes the awkward factor x.

Step 1. Choose the mirror substitution.

The interval is $[0,\pi]$, so use $x = \pi - u$. Call the integral K. Since $\sin(\pi-u) = \sin u$ and $\cos^2(\pi - u) = \cos^2 u$:

$$K = \int_0^{\pi} \frac{(\pi - u)\sin u}{1+\cos^2 u}\, du.$$

Step 2. Add to the original.

$$2K = \int_0^{\pi} \frac{x\sin x + (\pi - x)\sin x}{1+\cos^2 x}\, dx = \pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\, dx.$$

The troublesome x has gone.

Step 3. Integrate what is left.

Let $u = \cos x$, $du = -\sin x\, dx$; $x = 0 \to u = 1$, $x = \pi \to u = -1$:

$$\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\, dx = \int_1^{-1} \frac{-du}{1+u^2} = \int_{-1}^{1} \frac{du}{1+u^2} = \big[\arctan u\big]_{-1}^{1} = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}.$$

Step 4. Conclude.

$$2K = \pi \cdot \frac{\pi}{2} \quad \Longrightarrow \quad K = \frac{\pi^2}{4} \approx 2.467.$$

Answer. $\dfrac{\pi^2}{4}$

Common errors.

- Forgetting that $dx = -du$ flips the limits.
- Trying integration by parts: it goes round in circles.

How the marks are earned.

M1 Uses $x = \pi - u$

A1 $K = \int_0^{\pi} \frac{(\pi-u)\sin u}{1+\cos^2 u}\, du$

M1 $2K = \pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\, dx$

M1 $u = \cos x$ gives $\int_{-1}^{1} \frac{du}{1+u^2} = \frac{\pi}{2}$

A1 $K = \frac{\pi^2}{4}$

Question 8 *Logistic growth* 7 marks

The population, P, of a species of fish in a lake, t years after it was introduced, is modelled by

$$\frac{dP}{dt} = 0.1P\left(1 - \frac{P}{500}\right).$$

Initially there are 50 fish.

Solve the differential equation to show that

$$P = \frac{500}{1 + 9e^{-0.1t}}.$$

Solution.

The idea.

*This is the **logistic equation**: growth is fast when the population is small, and slows as it approaches the "carrying capacity" 500. Separating the variables needs partial fractions, and the logarithms then rearrange into the famous S-curve.*

Step 1. Tidy the right-hand side.

$$0.1P\left(1 - \frac{P}{500}\right) = \frac{P(500 - P)}{5000}.$$

So $\dfrac{dP}{dt} = \dfrac{P(500 - P)}{5000}$.

Step 2. Separate the variables.

$$\int \frac{5000}{P(500 - P)}\,dP = \int 1\,dt.$$

Step 3. Partial fractions.

$$\frac{5000}{P(500 - P)} \equiv \frac{A}{P} + \frac{B}{500 - P} \quad \Longrightarrow \quad 5000 \equiv A(500 - P) + BP.$$

$P = 0$: $5000 = 500A$, so $A = 10$. $P = 500$: $5000 = 500B$, so $B = 10$.

Step 4. Integrate.

$$\int \left(\frac{10}{P} + \frac{10}{500 - P}\right) dP = 10\ln P - 10\ln(500 - P).$$

↪ *The second term: $\int \frac{1}{500-P}\,dP = -\ln(500 - P)$, because of the minus sign inside. Moduli aren't needed since $0 < P < 500$ here.*

So

$$10\ln\frac{P}{500 - P} = t + c \quad \Longrightarrow \quad \ln\frac{P}{500 - P} = 0.1t + c'.$$

Step 5. Use the initial condition.

$t = 0$, $P = 50$: $c' = \ln\dfrac{50}{450} = \ln\dfrac{1}{9}$. So

$$\ln\frac{P}{500-P} = 0.1t + \ln\frac{1}{9} \quad\Longrightarrow\quad \frac{P}{500-P} = \frac{e^{0.1t}}{9}.$$

Step 6. Make P the subject.

$$9P = (500-P)e^{0.1t} \quad\Longrightarrow\quad P(9+e^{0.1t}) = 500e^{0.1t} \quad\Longrightarrow\quad P = \frac{500e^{0.1t}}{9+e^{0.1t}}.$$

Divide top and bottom by $e^{0.1t}$:

$$P = \frac{500}{1+9e^{-0.1t}}. \quad \blacksquare$$

Check $t=0$: $\frac{500}{10} = 50$ ✓.

> **Answer.** $P = \dfrac{500}{1+9e^{-0.1t}}$ (shown)

How the marks are earned.

M1 Separates: $\displaystyle\int \frac{5000}{P(500-P)}\,dP = \int dt$

M1 Partial fractions $\dfrac{A}{P} + \dfrac{B}{500-P}$

A1 $A = B = 10$

M1 $10\ln P - 10\ln(500-P) = t + c$

A1 $\displaystyle\ln\frac{P}{500-P} = 0.1t + c'$

M1 Uses $t=0,\ P=50$: $c' = \ln\frac{1}{9}$

A1* Rearranges to the given form

∎

Question 9 *Logistic growth: interpretation* 6 marks

The number of fish, P, in a lake t years after they were introduced satisfies

$$\frac{dP}{dt} = 0.1P\left(1 - \frac{P}{500}\right), \qquad P = \frac{500}{1 + 9e^{-0.1t}}.$$

(a) *Find the time taken for the population to reach 250.* *(2)*

(b) *State what happens to the population in the long term.* *(1)*

(c) *Show that the population is growing fastest when $P = 250$, and find this maximum rate of growth.* *(3)*

Solution.

The idea.

Each part reads something different from the model: solve $P = 250$ with logarithms, let $t \to \infty$ for the long term, and treat $\frac{dP}{dt}$ as a ***quadratic in P*** *to find where growth is fastest.*

Step 1. (a) Reach 250.

$$\frac{500}{1 + 9e^{-0.1t}} = 250 \quad \Longrightarrow \quad 1 + 9e^{-0.1t} = 2 \quad \Longrightarrow \quad e^{-0.1t} = \frac{1}{9}.$$

$$t = 10\ln 9 = 21.97\ldots \approx 22.0 \text{ years}.$$

Step 2. (b) Long term.

As $t \to \infty$, $9e^{-0.1t} \to 0$, so $P \to 500$. The population levels off at 500 fish, the lake's carrying capacity.

Step 3. (c) Growth rate as a function of P.

$$\frac{dP}{dt} = 0.1P - \frac{P^2}{5000},$$

a downward parabola in P with roots $P = 0$ and $P = 500$. Its vertex is halfway between the roots, at $P = 250$. (Or: $\frac{d}{dP}\left(0.1P - \frac{P^2}{5000}\right) = 0.1 - \frac{P}{2500} = 0 \Rightarrow P = 250$.)

Step 4. (c) The maximum rate.

$$\frac{dP}{dt} = 0.1 \times 250 \times \left(1 - \frac{250}{500}\right) = 25 \times 0.5 = 12.5 \text{ fish per year},$$

reached at $t \approx 22$ years, the steepest point of the S-curve.

Answer. (a) $10\ln 9 \approx 22.0$ years (b) $P \to 500$ (c) 12.5 fish per year

Common errors.

- $\int \frac{1}{500 - P}\,dP = \ln(500 - P)$ (sign error).
- Answering (c) by differentiating $P(t)$ with the quotient rule: possible, but far longer.

How the marks are earned.

M1 **(a)** $1 + 9e^{-0.1t} = 2$

A1 $t = 10\ln 9 \approx 22.0$ years

B1 **(b)** Tends to 500

M1 **(c)** $\dfrac{dP}{dt} = 0.1P - \dfrac{P^2}{5000}$, a quadratic in P

A1* Maximum at $P = 250$ (vertex or derivative)

A1 12.5 fish per year

∎

Question 10 *Vector geometry* 6 marks

> *The points A, B and C have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. A median of triangle ABC joins a vertex to the midpoint of the opposite side. Prove, using vectors, that the three medians of triangle ABC meet at a single point, and that this point divides each median in the ratio $2:1$.*

Solution.

The idea.

*For (a), write down the point two-thirds of the way along one median and notice the answer is **symmetric** in $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, so every median passes through it. For (b), intersecting lines must agree in **all three** coordinates.*

Step 1. One median.

Let M be the midpoint of BC: $\mathbf{m} = \frac{1}{2}(\mathbf{b}+\mathbf{c})$. The point G two-thirds of the way from A to M is

$$\mathbf{g} = \mathbf{a} + \tfrac{2}{3}(\mathbf{m}-\mathbf{a}) = \tfrac{1}{3}\mathbf{a} + \tfrac{2}{3}\mathbf{m} = \tfrac{1}{3}\mathbf{a} + \tfrac{1}{3}\mathbf{b} + \tfrac{1}{3}\mathbf{c}.$$

Step 2. Use the symmetry.

$\frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c})$ is unchanged if the letters are permuted. Repeating the calculation for the median from B (to the midpoint $\frac{1}{2}(\mathbf{a}+\mathbf{c})$) gives

$$\tfrac{1}{3}\mathbf{b} + \tfrac{2}{3}\cdot\tfrac{1}{2}(\mathbf{a}+\mathbf{c}) = \tfrac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c}),$$

and likewise for the median from C.

Step 3. Conclude.

All three medians pass through G with $\mathbf{g} = \frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c})$, and on each one G is two-thirds of the way from the vertex: it divides each median in the ratio $2:1$. ■

↪ *G is the **centroid**, the triangle's balance point.*

Answer. Proof as above.

How the marks are earned.

M1 Midpoint of BC: $\frac{1}{2}(\mathbf{b}+\mathbf{c})$
M1 Point $\frac{2}{3}$ of the way from A: $\mathbf{a} + \frac{2}{3}\left(\frac{1}{2}(\mathbf{b}+\mathbf{c}) - \mathbf{a}\right)$
A1 $= \frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c})$
M1 Same calculation (or symmetry) for the medians from B and C
A1 All give $\frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c})$
A1 Concludes: a common point, dividing each median $2:1$

∎

Question 11 *Intersecting lines* 8 marks

The lines l_1 and l_2 have equations

$$l_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \qquad l_2 : \mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$

(a) *Show that l_1 and l_2 intersect, and find the coordinates of their point of intersection.* ***(5)***

(b) *Find the acute angle between l_1 and l_2.* ***(3)***

Solution.

The idea.

*Intersecting lines must agree in **all three** coordinates: solve two of the equations for the parameters, then check the third. The angle between two lines is the angle between their direction vectors.*

Step 1. (a) Equate components.

At an intersection, both lines give the same point:

$$\begin{aligned} 1 + \lambda &= 3 + 2\mu \qquad (1) \\ 2 - \lambda &= 1 + \mu \qquad (2) \\ 3 + 2\lambda &= 6 + \mu \qquad (3) \end{aligned}$$

Step 2. (a) Solve two of them.

From (2): $\mu = 1 - \lambda$. Substitute into (1):

$$1 + \lambda = 3 + 2 - 2\lambda \quad \Longrightarrow \quad 3\lambda = 4 \quad \Longrightarrow \quad \lambda = \tfrac{4}{3}, \quad \mu = -\tfrac{1}{3}.$$

Step 3. (a) Check the third.

(3): left $3 + \frac{8}{3} = \frac{17}{3}$; right $6 - \frac{1}{3} = \frac{17}{3}$ ✓. All three equations hold, so the lines **do** intersect. (If the third had failed, the lines would be skew.)

Step 4. (a) The point.

Put $\lambda = \frac{4}{3}$ into l_1:

$$\left(1 + \tfrac{4}{3},\ 2 - \tfrac{4}{3},\ 3 + \tfrac{8}{3}\right) = \left(\tfrac{7}{3},\ \tfrac{2}{3},\ \tfrac{17}{3}\right).$$

Step 5. (b) Angle between the directions.

$$\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = 2 - 1 + 2 = 3, \qquad |\mathbf{d}_1| = |\mathbf{d}_2| = \sqrt{6}.$$

$$\cos\theta = \frac{3}{6} = \frac{1}{2} \quad \Longrightarrow \quad \theta = 60^\circ.$$

It's already acute. (If the cosine had been negative, the acute angle would be 180° minus the result.)

Answer. (a) $\left(\frac{7}{3}, \frac{2}{3}, \frac{17}{3}\right)$ (b) 60°

Common errors.

- Using the same letter for both parameters.

- Not checking the third equation, which is the whole point of "show that they intersect".

How the marks are earned.

M1 **(a)** Equates components

M1 Solves two equations

A1 $\lambda = \frac{4}{3},\ \mu = -\frac{1}{3}$

M1 Checks the third equation

A1 $\left(\frac{7}{3}, \frac{2}{3}, \frac{17}{3}\right)$

M1 **(b)** Scalar product of the direction vectors: $\mathbf{3}$

M1 $\cos\theta = \dfrac{3}{\sqrt{6}\sqrt{6}} = \frac{1}{2}$

A1 60°

∎

Question 12 *Inequalities and e* 6 marks

(a) *Prove that $e^x \geq 1 + x$ for all real x, with equality only when $x = 0$.* *(4)*

(b) *By choosing a suitable value of x, deduce that, for every positive integer n,* *(2)*

$$\left(1 + \frac{1}{n}\right)^n < e.$$

Solution.

The idea.

A single, simple inequality, $e^x \geq 1 + x$ (the curve lies above its tangent at 0), gives both an upper and a lower bound for e, from two clever substitutions: a famous squeeze.

Step 1. (a) Define a difference function.

Let $f(x) = e^x - 1 - x$. I want to show $f(x) \geq 0$ everywhere.

Step 2. (a) Find its minimum.

$f'(x) = e^x - 1$, which is 0 only at $x = 0$ (since $e^x = 1 \iff x = 0$). Also $f''(x) = e^x > 0$, so this is a minimum, and the **only** stationary point, so it is the global minimum.

Step 3. (a) Conclude.

$f(0) = 1 - 1 - 0 = 0$. So $f(x) \geq 0$ for all x, i.e. $e^x \geq 1 + x$, with equality only at $x = 0$. ■

↪ *Geometrically: $y = 1 + x$ is the tangent to $y = e^x$ at $(0, 1)$, and the curve is convex, so it lies above every tangent.*

Step 4. (b) Put x equal to 1 over n.

For $x = \frac{1}{n} \neq 0$ the inequality is strict:

$$e^{1/n} > 1 + \frac{1}{n}.$$

Both sides are positive, so raising to the power n keeps the order:

$$e > \left(1 + \frac{1}{n}\right)^n. \quad ■$$

Answer. Proofs as above.

How the marks are earned.

M1 **(a)** Considers $f(x) = e^x - 1 - x$
M1 $f'(x) = e^x - 1 = 0 \iff x = 0$
A1 Minimum (by $f'' > 0$ or sign change), $f(0) = 0$
A1 Concludes $f(x) \geq 0$, equality only at $x = 0$
M1 **(b)** $x = \frac{1}{n}$: $e^{1/n} > 1 + \frac{1}{n}$
A1* Raises to the power n (both sides positive)

■

Question 13 *Bounds for e* 7 marks

You may assume that $e^x \geq 1 + x$ for all real x, with equality only when $x = 0$, and that $\left(1 + \frac{1}{n}\right)^n < e$ for every positive integer n.

(a) *By choosing another suitable value of x, show that, for every positive integer n,* *(4)*

$$e < \left(1 + \frac{1}{n}\right)^{n+1}.$$

(b) *Hence show that $2.59 < e < 2.86$.* *(3)*

Solution.

The idea.

Substitute a cleverly chosen negative value, $x = -\frac{1}{n+1}$, into $e^x \geq 1 + x$. Then take reciprocals and raise to a power, keeping careful track of which way each inequality points.

Step 1. (a) Put x equal to minus 1 over n plus 1.

$$e^{-\frac{1}{n+1}} > 1 - \frac{1}{n+1} = \frac{n}{n+1}.$$

Step 2. (a) Take reciprocals.

Both sides are positive, and taking reciprocals **reverses** the inequality:

$$e^{\frac{1}{n+1}} < \frac{n+1}{n} = 1 + \frac{1}{n}.$$

Step 3. (a) Raise to the power n plus 1.

$$e < \left(1 + \frac{1}{n}\right)^{n+1}. \quad \blacksquare$$

Step 4. (b) Use n equal to 10.

$$(1.1)^{10} < e < (1.1)^{11}.$$

$1.1^{10} = 2.59374\ldots$ and $1.1^{11} = 2.85311\ldots$, so

$$2.59 < 2.5937\ldots < e < 2.8531\ldots < 2.86. \quad \blacksquare$$

(Indeed $e = 2.71828\ldots$.)

Answer. (b) $2.59 < e < 2.86$

Common errors.

- Forgetting that the reciprocal step reverses the inequality.
- Saying "$f'(0) = 0$ so it's a minimum" without checking it's a minimum.

How the marks are earned.

M1 **(a)** $x = -\dfrac{1}{n+1}$

A1 $e^{-1/(n+1)} > \dfrac{n}{n+1}$

M1 Takes reciprocals (reversing) and raises to the power $n+1$

A1* $e < \left(\dfrac{n+1}{n}\right)^{n+1}$

M1 **(b)** Uses $n = 10$

A1 $1.1^{10} = 2.5937\ldots > 2.59$

A1 $1.1^{11} = 2.8531\ldots < 2.86$

∎

Question 14 *Optimisation: the cylinder* 4 marks

> *A right circular cylinder is inscribed in a sphere of radius R: both circular edges of the cylinder lie on the sphere. Find the maximum possible volume of the cylinder, justifying that it is a maximum.*

Solution.

The idea.

By symmetry the cylinder is centred at the centre of the sphere. Pythagoras links its radius to its half-height, which leaves a one-variable optimisation.

Step 1. Set up the cylinder.

By symmetry the cylinder is centred at O. Let its height be $2y$ and its radius ρ. A top corner (ρ, y) lies on the sphere: $\rho^2 = R^2 - y^2$. So

$$V_c = \pi \rho^2 (2y) = 2\pi y(R^2 - y^2) = 2\pi(R^2 y - y^3), \qquad 0 < y < R.$$

Step 2. Maximise.

$$\frac{dV_c}{dy} = 2\pi(R^2 - 3y^2) = 0 \quad \Longrightarrow \quad y = \frac{R}{\sqrt{3}}.$$

$\frac{d^2V_c}{dy^2} = -12\pi y < 0$, so it's a maximum. Then $\rho^2 = R^2 - \frac{R^2}{3} = \frac{2R^2}{3}$ and

$$V_c = \pi \cdot \frac{2R^2}{3} \cdot \frac{2R}{\sqrt{3}} = \frac{4\pi R^3}{3\sqrt{3}} = \frac{4\sqrt{3}\,\pi R^3}{9}.$$

That's $\frac{1}{\sqrt{3}} \approx 57.7\%$ of the sphere, nearly twice as efficient as the cone.

Answer. $\dfrac{4\sqrt{3}\,\pi R^3}{9}$

Common errors.

- Using the full height $2y$ in Pythagoras instead of the half-height y.
- Forgetting to check that the stationary point is a maximum.

How the marks are earned.

M1 $V_c = 2\pi y(R^2 - y^2)$ with half-height y
M1 $\dfrac{dV_c}{dy} = 2\pi(R^2 - 3y^2) = 0$
A1 $y = \dfrac{R}{\sqrt{3}}$
A1 $V_c = \dfrac{4\sqrt{3}\,\pi R^3}{9}$

∎

Question 15 *Optimisation* 13 marks

A right circular cone is inscribed in a sphere of radius R: its vertex and the whole circumference of its base lie on the sphere. The height of the cone is h, where $0 < h < 2R$.

(a) *Show that the volume of the cone is* *(4)*

$$V = \frac{\pi}{3}\left(2Rh^2 - h^3\right).$$

(b) *Find, in terms of R, the maximum possible volume of the cone, justifying that it is a maximum.* *(7)*

(c) *Show that the largest cone fills exactly $\dfrac{8}{27}$ of the sphere.* *(2)*

A video explanation of this question is at lemmamathematics.com/papers/pure-4#q15.

Solution.

The idea.

Take a cross-section through the axis: a circle with an isosceles triangle inside. Pythagoras links the cone's base radius to its height. Then it's a one-variable optimisation. The video shows the cone changing shape while its volume is plotted.

Step 1. (a) Draw the cross-section.

Put the sphere's centre O at the origin and the cone's vertex at the top of the sphere, $(0, R)$. The base is a horizontal circle a distance h below the vertex, at height $R - h$, with radius r. A point on the base rim, $(r, R - h)$, lies on the sphere.

Step 2. (a) Pythagoras.

$$r^2 + (R - h)^2 = R^2 \quad \Longrightarrow \quad r^2 = R^2 - (R^2 - 2Rh + h^2) = 2Rh - h^2.$$

↪ *This works whether the base is above or below the centre ($h < R$ or $h > R$), since $(R - h)^2 = (h - R)^2$.*

Step 3. (a) The volume.

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(2Rh - h^2)h = \frac{\pi}{3}(2Rh^2 - h^3). \quad \blacksquare$$

Step 4. (b) Differentiate with respect to h.

R is a constant:

$$\frac{dV}{dh} = \frac{\pi}{3}(4Rh - 3h^2) = \frac{\pi}{3}h(4R - 3h).$$

Step 5. (b) Stationary point.

$h > 0$, so $\frac{dV}{dh} = 0$ gives $4R - 3h = 0$:

$$h = \frac{4R}{3}.$$

This lies in $0 < h < 2R$ ✓. The cone's base is a third of the radius **below** the centre.

Step 6. (b) Confirm it is a maximum.

$$\frac{d^2V}{dh^2} = \frac{\pi}{3}(4R - 6h); \qquad \text{at } h = \tfrac{4R}{3}: \ \frac{\pi}{3}(4R - 8R) = -\frac{4\pi R}{3} < 0.$$

So it is a maximum. (Also $V \to 0$ at both ends, $h \to 0$ and $h \to 2R$, so this single stationary point must be the global maximum.)

Step 7. (b) The maximum volume.

$$V = \frac{\pi}{3}\left(2R \cdot \frac{16R^2}{9} - \frac{64R^3}{27}\right) = \frac{\pi}{3}\left(\frac{96R^3 - 64R^3}{27}\right) = \frac{\pi}{3} \cdot \frac{32R^3}{27} = \frac{32\pi R^3}{81}.$$

Step 8. (c) Compare with the sphere.

$$\frac{32\pi R^3/81}{\frac{4}{3}\pi R^3} = \frac{32}{81} \times \frac{3}{4} = \frac{96}{324} = \frac{8}{27}. \quad \blacksquare$$

About 29.6%: the best cone fills less than a third of the sphere.

Answer. (b) $\dfrac{32\pi R^3}{81}$ at $h = \dfrac{4R}{3}$

Common errors.

- Taking the cone's slant height as R.
- Differentiating with respect to R instead of h.

How the marks are earned.

M1 (a) Cross-section: base radius r with $r^2 + (h-R)^2 = R^2$
A1 $r^2 = 2Rh - h^2$
M1 $V = \frac{1}{3}\pi r^2 h$
A1* $V = \frac{\pi}{3}(2Rh^2 - h^3)$
M1 (b) $\dfrac{dV}{dh} = \dfrac{\pi}{3}(4Rh - 3h^2)$
A1 Correct derivative
M1 Sets $= 0$ and solves
A1 $h = \dfrac{4R}{3}$
M1 $\dfrac{d^2V}{dh^2} = \dfrac{\pi}{3}(4R - 6h) < 0$ at $h = \frac{4R}{3}$
A1 Concludes maximum
A1 $V_{\max} = \dfrac{32\pi R^3}{81}$
M1 (c) $\dfrac{32\pi R^3/81}{4\pi R^3/3}$
A1* $= \dfrac{8}{27}$