LEMMA PRACTICE PAPER · MARK SCHEME

Level 5

Pure Mathematics

Paper 5 · STEP and olympiad style

GENERAL MARKING GUIDANCE

- All candidates must receive the same treatment: mark positively, and give credit for what is shown rather than penalising what is missing.
- **M** marks: method marks, for knowing and applying a correct method to the problem, even if there are small slips.
- **A** marks: accuracy marks, which can only be awarded if the relevant M mark has been earned.
- **B** marks: unconditional accuracy marks, independent of method.
- **dM**: a method mark that depends on the previous M mark. **ft**: follow through from an earlier error. *****: the answer is given in the question, so full working must be seen with no errors. **cao**: correct answer only.
- Correct alternative methods earn full credit. The model solutions show several of them.

QUESTION	SCHEME	MARKS	NOTES
1	$(x-y)(x+y)=2026$	**M1**	
	$(x+y)-(x-y)=2y$ is even, so the factors have the same parity	**M1**	
	Both odd $\Rightarrow$ product odd; both even $\Rightarrow$ product divisible by 4	**A1**	
	$2026=2\times1013$ is even but not divisible by 4	**B1**	
	Concludes no solutions	**A1**	
	Total for Question 1	**(5)**	

QUESTION	SCHEME	MARKS	NOTES
2	$d=x-y,\ e=x+y$ with $de=2025$	**M1**	
	$y>0\Rightarrow d<e$; all divisors are odd, so parities match	**M1**	
	$2025=3^4\times5^2$, with 15 divisors	**B1**	
	The 7 factor pairs with $d<e$	**A1**	
	$x=\frac{d+e}{2},\ y=\frac{e-d}{2}$	**M1**	
	At least four correct solutions	**A1**	
	All seven correct	**A1**	
	States there are no others (and $d=e=45$ gives $y=0$)	**A1**	
	Total for Question 2	**(8)**	

QUESTION	SCHEME	MARKS	NOTES
3	"Only if": a difference of squares is odd or a multiple of 4, as in (a)	**M1**	
	So $n \not\equiv 2 \pmod 4$	**A1**	
	"If", n odd: $n = 2k+1 = (k+1)^2 - k^2$	**M1**	
	Verified	**A1**	
	"If", $n = 4k$: $n = (k+1)^2 - (k-1)^2$	**M1**	
	Verified	**A1**	
	Notes $n \bmod 4 \in \{0, 1, 3\}$ covers every remaining case; conclusion	**A1**	
	Total for Question 3	**(7)**	

QUESTION	SCHEME	MARKS	NOTES
4 (a)	$\left(\sqrt{a} - \sqrt{b}\right)^2 \geq 0$	**M1**	
	Expands and rearranges to the result	**A1***	
	Equality iff $a = b$	**B1**	
(b)	$a + b \geq 2\sqrt{ab}$ and the two similar inequalities	**M1**	
	Multiplies (all sides positive)	**M1**	
	$\geq 8\sqrt{a^2b^2c^2}$	**A1**	
	$= 8abc$	**A1***	
	Total for Question 4	**(7)**	

QUESTION	SCHEME	MARKS	NOTES
5	$\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = (x+y+z)\left(\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z}\right)$	**M1**	
	$= 3 + \left(\dfrac{x}{y} + \dfrac{y}{x}\right) + \left(\dfrac{y}{z} + \dfrac{z}{y}\right) + \left(\dfrac{z}{x} + \dfrac{x}{z}\right)$	**A1**	
	Uses $t + \dfrac{1}{t} \geq 2$ for $t > 0$ (from (a))	**M1**	
	$\geq 3 + 2 + 2 + 2 = 9$	**A1***	
	Equality when $x = y = z = \frac{1}{3}$	**B1**	
	Total for Question 5	**(5)**	

QUESTION	SCHEME	MARKS	NOTES
6 (i)	Considers $\sum \left(x - \frac{1}{3}\right)^2 \geq 0$	**M1**	
	$\sum x^2 - \frac{2}{3}\sum x + \frac{1}{3} \geq 0$	**A1**	
	$\sum x^2 \geq \frac{1}{3}$	**A1***	

QUESTION	SCHEME	MARKS	NOTES
(ii)	$(x+y+z)^2 = \sum x^2 + 2\sum xy$	**M1**	
	$\sum xy = \frac{1}{2}\left(1 - \sum x^2\right) \leq \frac{1}{3}$	**A1***	
(iii)	Non-negative $\Rightarrow \sum xy \geq 0$	**M1**	
	$\sum x^2 = 1 - 2\sum xy \leq 1$	**A1**	
	Equality iff two of the variables are 0 (one of them is 1)	**B1**	
	Total for Question 6	**(8)**	

QUESTION	SCHEME	MARKS	NOTES
7	Writes $\sin^n x = \sin^{n-1} x \cdot \sin x$	**M1**	
	Parts: $\left[-\sin^{n-1} x \cos x\right] + (n-1)\int \sin^{n-2} x \cos^2 x \, dx$	**A1**	
	Boundary term is 0	**B1**	
	Uses $\cos^2 x = 1 - \sin^2 x$	**M1**	
	$I_n = (n-1)(I_{n-2} - I_n)$	**A1**	
	Rearranges	**M1**	
	$nI_n = (n-1)I_{n-2}$	**A1***	
	Total for Question 7	**(7)**	

QUESTION	SCHEME	MARKS	NOTES
8	$I_0 = \frac{\pi}{2}$	**B1**	
	$I_1 = 1$	**B1**	
	$I_4 = \frac{3\pi}{16}$	**B1**	
	$I_5 = \frac{8}{15}$	**B1**	
	Total for Question 8	**(4)**	

QUESTION	SCHEME	MARKS	NOTES
9 (a)	Multiplies the recurrence by I_{n-1}	**M1**	
	$nI_nI_{n-1} = (n-1)I_{n-1}I_{n-2}$	**A1**	
	So $a_n = nI_nI_{n-1}$ is constant	**M1**	
	$a_1 = I_1I_0 = \frac{\pi}{2}$	**A1**	
	Conclusion (induction stated)	**A1**	
(b)	$0 \leq \sin x \leq 1 \Rightarrow I_n \leq I_{n-1} \leq I_{n-2}$	**M1**	
	Divides by $I_{n-1} > 0$: $\dfrac{I_n}{I_{n-2}} \leq \dfrac{I_n}{I_{n-1}} \leq 1$	**A1**	
	Uses $\dfrac{I_n}{I_{n-2}} = \dfrac{n-1}{n}$	**M1**	

QUESTION	SCHEME	MARKS	NOTES
	Squeeze: limit is 1	**A1**	
	Total for Question 9	**(9)**	

QUESTION	SCHEME	MARKS	NOTES
10 (a)	$x_2 = \frac{3}{2}$	**B1**	
	$x_3 = \frac{17}{12}$	**B1**	
	$x_4 = \frac{577}{408}$	**B1**	
(b)	$x_{n+1} - \sqrt{2} = \dfrac{x_n^2 + 2}{2x_n} - \sqrt{2}$	**M1**	
	$= \dfrac{x_n^2 - 2\sqrt{2}x_n + 2}{2x_n}$	**A1**	
	Recognises a perfect square	**M1**	
	Result	**A1***	
	Total for Question 10	**(7)**	

QUESTION	SCHEME	MARKS	NOTES
11 (a)	All $x_n > 0$ (induction)	**M1**	
	$x_{n+1} - \sqrt{2} > 0$, strict since x_n is rational and $\neq \sqrt{2}$	**A1**	
	$x_{n+1} - x_n = \dfrac{2 - x_n^2}{2x_n}$	**M1**	
	< 0 for $n \geq 2$	**A1**	
(b)	$e_{n+1} = \dfrac{e_n^2}{2x_n} < \dfrac{e_n^2}{2\sqrt{2}} < \dfrac{e_n^2}{2.8}$	**M1**	
	Correct justification ($x_n > \sqrt{2}$, $2\sqrt{2} > 2.8$)	**A1**	
	$e_2 < 0.086$	**M1**	
	$e_3 < 0.0027$, $e_4 < 2.7 \times 10^{-6}$	**A1**	
	$e_5 < 2.7 \times 10^{-12} < 10^{-11}$	**A1**	
	Total for Question 11	**(9)**	

QUESTION	SCHEME	MARKS	NOTES
12	Checks $x_{n+1} = \dfrac{p_n^2 + 2q_n^2}{2p_nq_n}$ matches the recurrence	**M1**	
	$p_{n+1}^2 - 2q_{n+1}^2 = (p_n^2 + 2q_n^2)^2 - 8p_n^2q_n^2$	**M1**	
	$= (p_n^2 - 2q_n^2)^2$	**A1**	
	$n = 2$: $9 - 8 = 1$; induction completes	**A1**	
	Total for Question 12	**(4)**	

QUESTION	SCHEME	MARKS	NOTES
13 (a)	$\frac{dy}{dx} = 3x^2$, gradient $3p^2$	**M1**	
	$y - p^3 = 3p^2(x - p)$	**M1**	
	Simplifies to the result	**A1***	
(b)	$x^3 - 3p^2x + 2p^3 = 0$	**M1**	
	Notes $x = p$ is a repeated root (tangent)	**M1**	
	$(x - p)^2(x + 2p) = 0$	**A1**	
	$x = -2p$	**A1***	
	Total for Question 13	**(7)**	

QUESTION	SCHEME	MARKS	NOTES
14	Area $= \left\lvert \int_{-2p}^{p} \left(x^3 - 3p^2x + 2p^3\right) dx \right\rvert$	**M1**	
	Integrand $= (x - p)^2(x + 2p)$	**B1**	
	Substitution $u = x - p$ (or direct integration)	**M1**	
	$\int_{-3p}^{0} u^2(u + 3p)\, du$	**A1**	
	$\left[\frac{u^4}{4} + pu^3\right]$	**A1**	
	Evaluates: $-\frac{81p^4}{4} + 27p^4$	**M1**	
	$\frac{27p^4}{4}$, valid for either sign of p	**A1***	
	Total for Question 14	**(7)**	

QUESTION	SCHEME	MARKS	NOTES
15 (a)	The next tangent point has $x = -2p$, then $4p$, $-8p, \ldots$	**M1**	
	Next area $= \frac{27(2p)^4}{4} = 16 \times \frac{27p^4}{4}$	**A1**	
	Geometric with common ratio 16	**A1**	
(b)	Curve minus tangent at $x = t$ is $a(x - t)^2(x - s)$ with $2t + s = -\frac{b}{a}$	**M1**	
	Area $= \frac{\lvert a \rvert \lvert s - t \rvert^4}{12}$ and $s - k = -2(t - k)$ where $k = -\frac{b}{3a}$	**M1**	
	Distances from k double, so areas scale by $2^4 = 16$	**A1**	
	Total for Question 15	**(6)**	

TOTAL FOR PAPER: 100 MARKS