Name _____ Date _____ Score _____ / 100

LEMMA PRACTICE PAPER

Level 5

Pure Mathematics

Paper 5 · STEP and olympiad style

Paper reference **LP5/P** Time **3 hours**

TOTAL MARKS	100

You must have: pen, pencil, ruler, graph paper and a calculator.

INSTRUCTIONS

- Use black ink or ball-point pen; pencil may be used for diagrams.
- Answer **all** questions, in the spaces provided. If you need more space, continue on extra paper and label it clearly.
- Show all your working. An answer with no supporting method may get no marks.
- Give answers to 3 significant figures unless a question says otherwise, and give exact answers when asked.

INFORMATION

- There are **15** questions, and the paper is out of **100** marks.
- The marks for each part of a question are shown in brackets, for example **(3)**. Use them to judge how much to write.
- The questions are in roughly increasing order of difficulty.
- Difficulty: **STEP and olympiad style**. Long problems in the style of STEP and the MAT. Each needs an idea, not just a method; the parts guide you, but you must make the connections.

ADVICE

- Read each question carefully before you start.
- Keep an eye on the time: roughly one minute per mark.
- Check your answers if you have time at the end.
- The mark scheme, very detailed model solutions and video explanations are free at **lemmamathematics.com/papers/pure-5**.

1 This question is about writing integers as a difference of two squares, $x^2 - y^2$, where x and y are integers. **(5)**

Show that the equation $x^2 - y^2 = 2026$ has no integer solutions.

(Total for Question 1 is 5 marks)

2 Find all solutions of $x^2 - y^2 = 2025$ in **positive** integers x and y. **(8)**

(Total for Question 2 is 8 marks)

3 Prove that a positive integer n can be written as $x^2 - y^2$ with x, y integers if and only if n does not leave remainder 2 when divided by 4. **(7)**

(Total for Question 3 is 7 marks)

4 **(a)** Prove that, for positive real numbers a and b, **(3)**

$$\frac{a+b}{2} \geq \sqrt{ab},$$

and state when equality holds.

(b) Hence prove that, for positive real numbers a, b and c, **(4)**

$$(a+b)(b+c)(c+a) \geq 8abc.$$

(Total for Question 4 is 7 marks)

5 Positive real numbers x, y and z satisfy $x+y+z=1$. Prove that **(5)**

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \geq 9.$$

(Total for Question 5 is 5 marks)

6 Real numbers x, y and z (not necessarily positive) satisfy $x + y + z = 1$. (i) Prove that $x^2 + y^2 + z^2 \geq \frac{1}{3}$. (ii) Deduce that $xy + yz + zx \leq \frac{1}{3}$. (iii) Show further that if x, y and z are all non-negative, then $x^2 + y^2 + z^2 \leq 1$, and state when equality holds. **(8)**

(Total for Question 6 is 8 marks)

7 For integers $n \geq 0$, let **(7)**

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx.$$

Show that, for $n \geq 2$,

$$nI_n = (n-1)I_{n-2}.$$

(Total for Question 7 is 7 marks)

8 For integers $n \geq 0$, let $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$. You may assume that $nI_n = (n-1)I_{n-2}$ for $n \geq 2$. **(4)**

Find the exact values of I_0, I_1, I_4 and I_5.

(Total for Question 8 is 4 marks)

9 For integers $n \geq 0$, let $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$. You may assume that $nI_n = (n-1)I_{n-2}$ for $n \geq 2$, that $I_0 = \frac{\pi}{2}$ and that $I_1 = 1$.

(a) Show that $nI_nI_{n-1} = \dfrac{\pi}{2}$ for all $n \geq 1$. **(5)**

(b) By considering I_n, I_{n-1} and I_{n-2}, prove that $\dfrac{I_n}{I_{n-1}} \to 1$ as $n \to \infty$. **(4)**

(Total for Question 9 is 9 marks)

10 The sequence $x_1, x_2, x_3, \ldots$ is defined by

$$x_1 = 1, \qquad x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right).$$

(a) Find x_2, x_3 and x_4 as fractions. **(3)**

(b) Show that **(4)**

$$x_{n+1} - \sqrt{2} = \frac{\left(x_n - \sqrt{2}\right)^2}{2x_n}.$$

(Total for Question 10 is 7 marks)

11 The sequence $x_1, x_2, \ldots$ is defined by $x_1 = 1$ and $x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$. You may assume that

$$x_{n+1} - \sqrt{2} = \frac{\left(x_n - \sqrt{2}\right)^2}{2x_n}$$

and that $x_2 = \frac{3}{2}$.

(a) Deduce that $x_n > \sqrt{2}$ for all $n \geq 2$, and that the sequence is decreasing from x_2 onwards. **(4)**

(b) Let $e_n = x_n - \sqrt{2}$. Show that $e_{n+1} < \frac{e_n^2}{2.8}$ for $n \geq 2$, and hence, without evaluating x_5, show that $0 < x_5 - \sqrt{2} < 10^{-11}$. **(5)**

(Total for Question 11 is 9 marks)

12 The sequence $x_1, x_2, \ldots$ is defined by $x_1 = 1$ and $x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$. **(4)**

Write $x_n = \frac{p_n}{q_n}$, where $p_1 = q_1 = 1$, $p_{n+1} = p_n^2 + 2q_n^2$ and $q_{n+1} = 2p_n q_n$. Prove that $p_n^2 - 2q_n^2 = 1$ for all $n \geq 2$.

(Total for Question 12 is 4 marks)

13 The point P has coordinates (p, p^3), where $p \neq 0$, and lies on the curve C with equation $y = x^3$.

(a) Show that the tangent to C at P has equation $y = 3p^2x - 2p^3$. **(3)**

(b) Show that this tangent meets C again at the point Q with x-coordinate $-2p$. **(4)**

(Total for Question 13 is 7 marks)

14 The tangent to the curve $y = x^3$ at the point $P(p, p^3)$, where $p \neq 0$, has equation $y = 3p^2x - 2p^3$, and it meets the curve again at the point with x-coordinate $-2p$. **(7)**

Show that the area of the region enclosed between C and the tangent at P is $\dfrac{27p^4}{4}$.

(Total for Question 14 is 7 marks)

15 For $p \neq 0$, the tangent to $y = x^3$ at $P(p, p^3)$ meets the curve again at the point with x-coordinate $-2p$, and the area enclosed between the curve and this tangent is $\frac{27p^4}{4}$. The tangent at Q (the second point) meets the curve again at R, the tangent at R meets it again at S, and so on.

(a) Show that the areas enclosed between C and successive tangents form a geometric sequence, and state its common ratio. **(3)**

(b) Show that the same common ratio arises for **every** cubic curve $y = ax^3 + bx^2 + cx + d$, where $a \neq 0$. **(3)**

(Total for Question 15 is 6 marks)

TOTAL FOR PAPER IS 100 MARKS

END OF PAPER