Pure Mathematics

Paper 5 · STEP and olympiad style: model solutions

LP5/P · Level 5

How to use these solutions. Try each question under timed conditions first, then mark it with the mark scheme. Read the full solution for every question, even the ones you got right: they often show a faster method, a check, or a common error to avoid. Each solution begins with *the idea*, then works through numbered steps; remarks marked ↪ explain *why*. Each ends with the boxed answer and how the marks are earned.

Question 1 *Number theory* 5 marks

> *This question is about writing integers as a difference of two squares, $x^2 - y^2$, where x and y are integers.*
>
> *Show that the equation $x^2 - y^2 = 2026$ has no integer solutions.*

Solution.

The idea.

$x^2 - y^2 = (x - y)(x + y)$ *turns a question about squares into a question about* ***factor pairs****. The key observation: $x - y$ and $x + y$ always have the* ***same parity*** *(both odd or both even), because they differ by $2y$.*

Step 1. Factorise.

$$x^2 - y^2 = (x - y)(x + y) = 2026.$$

Step 2. The parity observation.

$(x + y) - (x - y) = 2y$ is even, so $x + y$ and $x - y$ are either **both odd** or **both even**.

- Both odd: their product is odd.
- Both even: each has a factor of 2, so the product is a multiple of 4.

Step 3. Apply it to 2026.

$2026 = 2 \times 1013$ (and 1013 is odd). So 2026 is even (not "both odd"), but $\frac{2026}{4} = 506.5$ is not an integer (not "both even"). Neither case is possible, so there are no integer solutions. ■

Answer. No integer solutions.

How the marks are earned.

M1 $(x-y)(x+y) = 2026$

M1 $(x+y)-(x-y) = 2y$ is even, so the factors have the same parity

A1 Both odd ⇒ product odd; both even ⇒ product divisible by 4

B1 $2026 = 2 \times 1013$ is even but not divisible by 4

A1 Concludes no solutions

■

Question 2 *Number theory: factor pairs* 8 marks

> *Find all solutions of* $x^2 - y^2 = 2025$ *in* ***positive*** *integers* x *and* y*.*

Solution.

The idea.
Write $x^2 - y^2 = (x-y)(x+y)$*: every solution comes from a factor pair of 2025, and every suitable factor pair gives a solution.*

Step 1. Turn it into factor pairs.
Let $d = x - y$ and $e = x + y$. Then $de = 2025$, and

$$x = \frac{d+e}{2}, \qquad y = \frac{e-d}{2}.$$

For positive x and y: $e > d > 0$. And since 2025 is odd, all its divisors are odd, so $d + e$ and $e - d$ are automatically even: every factor pair works.

Step 2. List the divisors.
$2025 = 45^2 = 3^4 \times 5^2$, which has $(4+1)(2+1) = 15$ divisors:

$$1, 3, 5, 9, 15, 25, 27, 45, 75, 81, 135, 225, 405, 675, 2025.$$

Pairs with $d < e$ (excluding $d = e = 45$, which gives $y = 0$):

$$(1, 2025), (3, 675), (5, 405), (9, 225), (15, 135), (25, 81), (27, 75).$$

Step 3. Compute each solution.

d	e	$x = \frac{d+e}{2}$	$y = \frac{e-d}{2}$
1	2025	1013	1012
3	675	339	336
5	405	205	200
9	225	117	108
15	135	75	60
25	81	53	28
27	75	51	24

Spot check: $53^2 - 28^2 = 2809 - 784 = 2025$ ✓; $75^2 - 60^2 = 5625 - 3600 = 2025$ ✓.

Step 4. There are no others.
Every solution gives a factor pair (d, e) with $d < e$, and every such pair gives exactly one solution. So these seven are all of them.

Answer. $(x, y) =$
$(1013, 1012), (339, 336), (205, 200), (117, 108), (75, 60), (53, 28), (51, 24)$

Common errors.

- Forgetting the parity argument and only checking factor pairs by hand.
- Including $(45, 0)$, when the question asks for positive integers.

How the marks are earned.

M1 $d = x - y$, $e = x + y$ with $de = 2025$
M1 $y > 0 \Rightarrow d < e$; all divisors are odd, so parities match
B1 $2025 = 3^4 \times 5^2$, with 15 divisors
A1 The 7 factor pairs with $d < e$
M1 $x = \frac{d+e}{2}$, $y = \frac{e-d}{2}$
A1 At least four correct solutions
A1 All seven correct
A1 States there are no others (and $d = e = 45$ gives $y = 0$)

∎

Question 3 *Number theory: differences of squares* 7 marks

> *Prove that a positive integer n can be written as $x^2 - y^2$ with x, y integers if and only if n does not leave remainder 2 when divided by 4.*

Solution.

The idea.

*An "if and only if" needs **both** directions. For "only if", use the fact that $x - y$ and $x + y$ have the same parity. For "if", write odd numbers and multiples of 4 as differences of two squares explicitly.*

Step 1. The "only if" direction.

Suppose $n = x^2 - y^2$. Since $(x + y) - (x - y) = 2y$ is even, $x - y$ and $x + y$ have the same parity, so n is either odd or a multiple of 4. In particular n is **not** $\equiv 2 \pmod 4$.

Step 2. The "if" direction: odd n.

If $n = 2k + 1$, then

$$(k+1)^2 - k^2 = k^2 + 2k + 1 - k^2 = 2k + 1 = n. \checkmark$$

Every odd number is the difference of two consecutive squares.

Step 3. The "if" direction: multiples of 4.

If $n = 4k$, then

$$(k+1)^2 - (k-1)^2 = (k^2 + 2k + 1) - (k^2 - 2k + 1) = 4k = n. \checkmark$$

Step 4. Conclude.

A positive integer leaves remainder 0, 1, 2 or 3 on division by 4. Remainders 1 and 3 mean odd, and remainder 0 means a multiple of 4: all are differences of two squares. Remainder 2 never is. ■

↪ *A small surprise: 2026 can't be written this way, but its neighbours 2025 and 2027 both can.*

Answer. Proof as above.

Common errors.

- Proving only one direction of the "if and only if".
- Forgetting the parity argument, which is the whole reason $4k + 2$ fails.

How the marks are earned.

M1 "Only if": a difference of squares is odd or a multiple of 4, as in (a)

A1 So $n \not\equiv 2 \pmod 4$

M1 "If", n odd: $n = 2k + 1 = (k+1)^2 - k^2$

A1 Verified

M1 "If", $n = 4k$: $n = (k+1)^2 - (k-1)^2$

A1 Verified

A1 Notes $n \bmod 4 \in \{0, 1, 3\}$ covers every remaining case; conclusion

∎

Question 4 *Inequalities* 7 marks

(a) *Prove that, for positive real numbers a and b,* *(3)*

$$\frac{a+b}{2} \geq \sqrt{ab},$$

and state when equality holds.

(b) *Hence prove that, for positive real numbers a, b and c,* *(4)*

$$(a+b)(b+c)(c+a) \geq 8abc.$$

Solution.

The idea.

Everything grows from one fact: ***a square is never negative****. Part (a) is that fact in disguise, and (b) and (c) build on it. In (d), the trick is to compare each variable with the average, $\frac{1}{3}$.*

Step 1. (a) Start from a square.

$\sqrt{a}$ and $\sqrt{b}$ are real since $a, b > 0$, so

$$\left(\sqrt{a}-\sqrt{b}\right)^2 \geq 0 \quad\Longrightarrow\quad a - 2\sqrt{ab} + b \geq 0 \quad\Longrightarrow\quad \frac{a+b}{2} \geq \sqrt{ab}. \quad \blacksquare$$

Equality holds exactly when $\sqrt{a} = \sqrt{b}$, i.e. $a = b$.

↪ *This is the AM–GM inequality: the arithmetic mean is at least the geometric mean.*

Step 2. (b) Three copies of (a).

$$a+b \geq 2\sqrt{ab}, \qquad b+c \geq 2\sqrt{bc}, \qquad c+a \geq 2\sqrt{ca}.$$

Step 3. (b) Multiply.

All six quantities are positive, so multiplying the inequalities is allowed:

$$(a+b)(b+c)(c+a) \geq 8\sqrt{ab}\sqrt{bc}\sqrt{ca} = 8\sqrt{a^2b^2c^2} = 8abc. \quad \blacksquare$$

↪ *Multiplying inequalities needs positivity: $-3 < 1$ and $-3 < 1$, but $9 > 1$.*

Answer. Proofs as above.

How the marks are earned.

M1 **(a)** $\left(\sqrt{a}-\sqrt{b}\right)^2 \geq 0$

A1* Expands and rearranges to the result

B1 Equality iff $a = b$

M1 **(b)** $a + b \geq 2\sqrt{ab}$ and the two similar inequalities

M1 Multiplies (all sides positive)

A1 $\geq 8\sqrt{a^2b^2c^2}$

A1* $= 8abc$

∎

Question 5 *Inequalities: reciprocals* 5 marks

> *Positive real numbers x, y and z satisfy $x+y+z=1$. Prove that*
>
> $$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\geq 9.$$

Solution.

The idea.

Multiply by 1 in the form $x+y+z$, expand, and pair each fraction with its reciprocal: each pair is at least 2.

Step 1. Multiply by 1 in disguise.

Since $x+y+z=1$,

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right).$$

Step 2. Expand into pairs.

The nine products are $\frac{x}{x}=1$ (three times) and the six mixed ratios:

$$=3+\left(\frac{x}{y}+\frac{y}{x}\right)+\left(\frac{y}{z}+\frac{z}{y}\right)+\left(\frac{z}{x}+\frac{x}{z}\right).$$

Step 3. Each pair is at least 2.

For $t>0$, the AM–GM inequality $\frac{a+b}{2}\geq\sqrt{ab}$ with $a=t$, $b=\frac{1}{t}$ gives $\frac{t+1/t}{2}\geq 1$, i.e. $t+\frac{1}{t}\geq 2$. Apply it to $t=\frac{x}{y}$, $\frac{y}{z}$, $\frac{z}{x}$:

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\geq 3+2+2+2=9. \quad \blacksquare$$

Equality needs $x=y=z$, so $x=y=z=\frac{1}{3}$, where the sum is exactly $3+3+3=9$.

> **Answer.** Proof as above; equality at $x=y=z=\frac{1}{3}$.

Common errors.

- Multiplying inequalities without noting that everything is positive.

How the marks are earned.

M1 $\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=(x+y+z)\left(\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}\right)$

A1 $=3+\left(\dfrac{x}{y}+\dfrac{y}{x}\right)+\left(\dfrac{y}{z}+\dfrac{z}{y}\right)+\left(\dfrac{z}{x}+\dfrac{x}{z}\right)$

M1 Uses $t+\dfrac{1}{t}\geq 2$ for $t>0$ (from (a))

A1* $\geq 3+2+2+2=9$

B1 Equality when $x=y=z=\frac{1}{3}$

∎

Question 6 *Inequalities: sums of squares* 8 marks

> *Real numbers x, y and z (not necessarily positive) satisfy $x+y+z=1$. (i) Prove that $x^2+y^2+z^2 \geq \frac{1}{3}$. (ii) Deduce that $xy+yz+zx \leq \frac{1}{3}$. (iii) Show further that if x, y and z are all non-negative, then $x^2+y^2+z^2 \leq 1$, and state when equality holds.*

Solution.

The idea.

Compare each variable with the average, $\frac{1}{3}$: a sum of squares is never negative. Then the identity $(x+y+z)^2 = x^2+y^2+z^2+2(xy+yz+zx)$ links the two sums.

Step 1. (i) Compare with the average.

For any reals,

$$\left(x-\tfrac{1}{3}\right)^2+\left(y-\tfrac{1}{3}\right)^2+\left(z-\tfrac{1}{3}\right)^2 \geq 0.$$

Expand:

$$(x^2+y^2+z^2)-\tfrac{2}{3}(x+y+z)+3\cdot\tfrac{1}{9} \geq 0 \quad\Longrightarrow\quad x^2+y^2+z^2-\tfrac{2}{3}+\tfrac{1}{3} \geq 0.$$

So $x^2+y^2+z^2 \geq \frac{1}{3}$. ■ (Equality iff $x=y=z=\frac{1}{3}$.)

Step 2. (ii) The square of the sum.

$$(x+y+z)^2 = x^2+y^2+z^2+2(xy+yz+zx) = 1.$$

So

$$xy+yz+zx = \frac{1-(x^2+y^2+z^2)}{2} \leq \frac{1-\frac{1}{3}}{2} = \frac{1}{3}. \quad \blacksquare$$

Step 3. (iii) Non-negative case.

If $x,y,z \geq 0$, every product $xy, yz, zx \geq 0$, so

$$x^2+y^2+z^2 = 1-2(xy+yz+zx) \leq 1.$$

Equality needs $xy=yz=zx=0$, so at least two of the variables are zero, and the third is then 1: $(1,0,0)$ in some order.

↪ *Together: on the triangle $x+y+z=1$, $x,y,z \geq 0$, the sum of squares ranges from $\frac{1}{3}$ (centre) to 1 (corners).*

> **Answer.** Proofs as above; equality in (iii) at $(1,0,0)$ and its permutations.

Common errors.

- Assuming the variables are positive in (i) and (ii), which don't need it.

- In (iii), forgetting to say when equality holds.

How the marks are earned.

M1 **(i)** Considers $\sum \left(x - \frac{1}{3}\right)^2 \geq 0$
A1 $\sum x^2 - \frac{2}{3}\sum x + \frac{1}{3} \geq 0$
A1* $\sum x^2 \geq \frac{1}{3}$
M1 **(ii)** $(x+y+z)^2 = \sum x^2 + 2\sum xy$
A1* $\sum xy = \frac{1}{2}\left(1 - \sum x^2\right) \leq \frac{1}{3}$
M1 **(iii)** Non-negative $\Rightarrow \sum xy \geq 0$
A1 $\sum x^2 = 1 - 2\sum xy \leq 1$
B1 Equality iff two of the variables are 0 (one of them is 1)

∎

Question 7 *Reduction formulae* 7 marks

For integers $n \geq 0$, let

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx.$$

Show that, for $n \geq 2$,

$$nI_n = (n-1)I_{n-2}.$$

Solution.

The idea.
*Integration by parts lowers the power by two. That recurrence lets you compute any I_n, and it also hides an **invariant**, nI_nI_{n-1}, which never changes. Part (d) is the key step in the famous Wallis product for π.*

Step 1. Split off one sine.

$$I_n = \int_0^{\pi/2} \sin^{n-1} x \, \sin x \, dx.$$

Take $u = \sin^{n-1} x$, $\frac{dv}{dx} = \sin x$, so $\frac{du}{dx} = (n-1)\sin^{n-2} x \cos x$ and $v = -\cos x$.

Step 2. Integrate by parts.

$$I_n = \Big[-\sin^{n-1} x \cos x \Big]_0^{\pi/2} + (n-1) \int_0^{\pi/2} \sin^{n-2} x \cos^2 x \, dx.$$

The boundary term vanishes: at $\frac{\pi}{2}$, $\cos = 0$; at 0, $\sin = 0$ (as $n - 1 \geq 1$).

Step 3. Back to sines only.
$\cos^2 x = 1 - \sin^2 x$:

$$I_n = (n-1) \int_0^{\pi/2} \left(\sin^{n-2} x - \sin^n x \right) dx = (n-1)(I_{n-2} - I_n).$$

Step 4. Rearrange.

$$I_n + (n-1)I_n = (n-1)I_{n-2} \quad \Longrightarrow \quad nI_n = (n-1)I_{n-2}. \quad \blacksquare$$

$\hookrightarrow$ *I_n appeared on both sides: that's not circular, just an equation to solve for I_n.*

Answer. Proof as above.

Common errors.

- Not explaining why the boundary term is zero.
- Forgetting that I_n appears on both sides, and stopping before solving for it.

How the marks are earned.

M1 Writes $\sin^n x = \sin^{n-1} x \cdot \sin x$

A1 Parts: $\left[-\sin^{n-1} x \cos x\right] + (n-1)\int \sin^{n-2} x \cos^2 x \, dx$

B1 Boundary term is 0

M1 Uses $\cos^2 x = 1 - \sin^2 x$

A1 $I_n = (n-1)(I_{n-2} - I_n)$

M1 Rearranges

A1* $nI_n = (n-1)I_{n-2}$

∎

Question 8 *Reduction formulae: values* 4 marks

For integers $n \geq 0$, let $I_n = \displaystyle\int_0^{\frac{\pi}{2}} \sin^n x \, dx$. You may assume that $nI_n = (n-1)I_{n-2}$ for $n \geq 2$.

Find the exact values of I_0, I_1, I_4 and I_5.

Solution.

The idea.

Two direct integrations start the ladder. The recurrence then climbs two rungs at a time: even n from I_0, odd n from I_1.

Step 1. The two starting values.

$$I_0 = \int_0^{\pi/2} 1 \, dx = \frac{\pi}{2}, \qquad I_1 = \int_0^{\pi/2} \sin x \, dx = \Big[-\cos x \Big]_0^{\pi/2} = 1.$$

Step 2. Climb the ladder.
$I_n = \frac{n-1}{n} I_{n-2}$:

$$I_2 = \tfrac{1}{2} I_0 = \tfrac{\pi}{4}, \qquad I_4 = \tfrac{3}{4} I_2 = \tfrac{3\pi}{16};$$

$$I_3 = \tfrac{2}{3} I_1 = \tfrac{2}{3}, \qquad I_5 = \tfrac{4}{5} I_3 = \tfrac{8}{15}.$$

Even n give multiples of π; odd n give rationals.

Answer. $I_0 = \frac{\pi}{2}$, $I_1 = 1$, $I_4 = \frac{3\pi}{16}$, $I_5 = \frac{8}{15}$

Common errors.

- Using $I_n = \frac{n-1}{n} I_{n-1}$ (the recurrence steps by **two**).

How the marks are earned.

B1 $I_0 = \frac{\pi}{2}$
B1 $I_1 = 1$
B1 $I_4 = \frac{3\pi}{16}$
B1 $I_5 = \frac{8}{15}$

∎

Question 9 *Reduction formulae: an invariant* 9 marks

For integers $n \geq 0$, let $I_n = \displaystyle\int_0^{\frac{\pi}{2}} \sin^n x \, dx$. You may assume that $nI_n = (n-1)I_{n-2}$ for $n \geq 2$, that $I_0 = \frac{\pi}{2}$ and that $I_1 = 1$.

(a) *Show that $nI_nI_{n-1} = \dfrac{\pi}{2}$ for all $n \geq 1$.* *(5)*

(b) *By considering I_n, I_{n-1} and I_{n-2}, prove that $\dfrac{I_n}{I_{n-1}} \to 1$ as $n \to \infty$.* *(4)*

Solution.

The idea.

Multiplying the recurrence by I_{n-1} reveals a quantity that never changes. For the limit, squeeze $\frac{I_n}{I_{n-1}}$ between two things that both tend to 1.

Step 1. (a) The invariant.

Multiply $nI_n = (n-1)I_{n-2}$ by I_{n-1}:

$$nI_nI_{n-1} = (n-1)I_{n-1}I_{n-2}.$$

With $a_n = nI_nI_{n-1}$ this says $a_n = a_{n-1}$ for every $n \geq 2$.

Step 2. (a) Evaluate the constant.

By induction, $a_n = a_1$ for all $n \geq 1$, and

$$a_1 = 1 \cdot I_1 \cdot I_0 = \frac{\pi}{2}.$$

So $nI_nI_{n-1} = \frac{\pi}{2}$. ■ **Check** (with $I_4 = \frac{3\pi}{16}$ and $I_5 = \frac{8}{15}$): $5I_5I_4 = 5 \cdot \frac{8}{15} \cdot \frac{3\pi}{16} = \frac{\pi}{2}$ ✓.

Step 3. (b) The integrals decrease.

On $\left[0, \frac{\pi}{2}\right]$, $0 \leq \sin x \leq 1$, so $\sin^n x \leq \sin^{n-1} x \leq \sin^{n-2} x$. Integrating:

$$0 < I_n \leq I_{n-1} \leq I_{n-2}.$$

Step 4. (b) Squeeze.

Dividing $I_n \leq I_{n-1}$ by I_{n-1}, and $I_{n-1} \leq I_{n-2}$ (multiplied by I_n) by $I_{n-1}I_{n-2}$, gives

$$\frac{I_n}{I_{n-2}} \leq \frac{I_n}{I_{n-1}} \leq 1.$$

(The left inequality is $I_{n-1} \leq I_{n-2}$ rearranged.) By the recurrence, $\frac{I_n}{I_{n-2}} = \frac{n-1}{n} \to 1$, so by the squeeze

$$\frac{I_n}{I_{n-1}} \to 1. \quad \blacksquare$$

Step 5. Where this leads.

Combining (a) and (b) with the values from the previous question gives **Wallis's product**,

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots,$$

and (a) shows $I_n \approx \sqrt{\frac{\pi}{2n}}$ for large n.

Answer. (a) $nI_nI_{n-1} = \frac{\pi}{2}$ (b) $\dfrac{I_n}{I_{n-1}} \to 1$

Common errors.

- Dividing inequalities without saying that the integrals are positive.
- Claiming $\frac{I_n}{I_{n-1}} \to 1$ "because both tend to 0": that alone proves nothing.

How the marks are earned.

M1 **(a)** Multiplies the recurrence by I_{n-1}
A1 $nI_nI_{n-1} = (n-1)I_{n-1}I_{n-2}$
M1 So $a_n = nI_nI_{n-1}$ is constant
A1 $a_1 = I_1I_0 = \frac{\pi}{2}$
A1 Conclusion (induction stated)
M1 **(b)** $0 \le \sin x \le 1 \Rightarrow I_n \le I_{n-1} \le I_{n-2}$
A1 Divides by $I_{n-1} > 0$: $\dfrac{I_n}{I_{n-2}} \le \dfrac{I_n}{I_{n-1}} \le 1$
M1 Uses $\dfrac{I_n}{I_{n-2}} = \dfrac{n-1}{n}$
A1 Squeeze: limit is 1

∎

Question 10 *Iteration and convergence* 7 marks

The sequence $x_1, x_2, x_3, \ldots$ is defined by

$$x_1 = 1, \qquad x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right).$$

(a) *Find x_2, x_3 and x_4 as fractions.* *(3)*

(b) *Show that* *(4)*

$$x_{n+1} - \sqrt{2} = \frac{\left(x_n - \sqrt{2}\right)^2}{2x_n}.$$

Solution.

The idea.

*This is Newton's method (and the Babylonian method) for $\sqrt{2}$. Part (b) is the heart of it: the new error is the **square** of the old error, divided by roughly $2\sqrt{2}$. So the number of correct digits roughly **doubles** every step.*

Step 1. (a) Iterate.

$$x_2 = \tfrac{1}{2}(1+2) = \tfrac{3}{2}, \qquad x_3 = \tfrac{1}{2}\left(\tfrac{3}{2} + \tfrac{4}{3}\right) = \tfrac{1}{2}\cdot\tfrac{17}{6} = \tfrac{17}{12},$$

$$x_4 = \tfrac{1}{2}\left(\tfrac{17}{12} + \tfrac{24}{17}\right) = \tfrac{1}{2}\cdot\tfrac{289+288}{204} = \tfrac{577}{408}.$$

($577/408 = 1.41421568\ldots$, while $\sqrt{2} = 1.41421356\ldots$.)

Step 2. (b) Combine into one fraction.

$$x_{n+1} = \frac{x_n^2 + 2}{2x_n}, \qquad x_{n+1} - \sqrt{2} = \frac{x_n^2 + 2 - 2\sqrt{2}\,x_n}{2x_n}.$$

Step 3. (b) Spot the square.

$x_n^2 - 2\sqrt{2}\,x_n + 2 = \left(x_n - \sqrt{2}\right)^2$, because $(\sqrt{2})^2 = 2$. So

$$x_{n+1} - \sqrt{2} = \frac{\left(x_n - \sqrt{2}\right)^2}{2x_n}. \quad \blacksquare$$

Answer. $x_2 = \frac{3}{2}$, $x_3 = \frac{17}{12}$, $x_4 = \frac{577}{408}$

How the marks are earned.

B1 (a) $x_2 = \frac{3}{2}$

B1 $x_3 = \frac{17}{12}$

B1 $x_4 = \frac{577}{408}$

M1 (b) $x_{n+1} - \sqrt{2} = \dfrac{x_n^2 + 2}{2x_n} - \sqrt{2}$

A1 $= \dfrac{x_n^2 - 2\sqrt{2}x_n + 2}{2x_n}$

M1 Recognises a perfect square

A1* Result

∎

Question 11 *Iteration: monotonicity and errors* 9 marks

The sequence $x_1, x_2, \ldots$ is defined by $x_1 = 1$ and $x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$. You may assume that

$$x_{n+1} - \sqrt{2} = \frac{\left(x_n - \sqrt{2}\right)^2}{2x_n}$$

and that $x_2 = \frac{3}{2}$.

(a) *Deduce that $x_n > \sqrt{2}$ for all $n \geq 2$, and that the sequence is decreasing from x_2 onwards.* *(4)*

(b) *Let $e_n = x_n - \sqrt{2}$. Show that $e_{n+1} < \dfrac{e_n^2}{2.8}$ for $n \geq 2$, and hence, without evaluating x_5, show that $0 < x_5 - \sqrt{2} < 10^{-11}$.* *(5)*

Solution.

The idea.

*The given identity says the new error is the **square** of the old error divided by $2x_n$. Everything follows from it: the terms stay above $\sqrt{2}$, they decrease, and the error shrinks quadratically.*

Step 1. (a) Everything stays positive.

$x_1 = 1 > 0$, and if $x_n > 0$ then $x_{n+1} = \frac{x_n^2+2}{2x_n} > 0$. By induction every $x_n > 0$.

Step 2. (a) Above the square root.

For $n \geq 1$, the given identity gives $x_{n+1} - \sqrt{2} \geq 0$, as a square divided by a positive number. It's strictly positive because each x_n is a fraction (by induction), so $x_n \neq \sqrt{2}$, which is irrational. So $x_n > \sqrt{2}$ for all $n \geq 2$.

Step 3. (a) Decreasing.

$$x_{n+1} - x_n = \frac{x_n^2 + 2}{2x_n} - x_n = \frac{2 - x_n^2}{2x_n}.$$

For $n \geq 2$, $x_n > \sqrt{2}$ means $x_n^2 > 2$, so this is negative: $x_{n+1} < x_n$. ■

Step 4. (b) The error bound.

From the given identity, $e_{n+1} = \dfrac{e_n^2}{2x_n}$. For $n \geq 2$, $x_n > \sqrt{2}$, so $2x_n > 2\sqrt{2} = 2.828\ldots > 2.8$:

$$e_{n+1} < \frac{e_n^2}{2.8}.$$

Step 5. (b) Chain it from e_2.

$e_2 = 1.5 - \sqrt{2}$, and $\sqrt{2} > 1.414$, so $e_2 < 0.086$. Then

$$e_3 < \frac{0.086^2}{2.8} = \frac{0.007396}{2.8} < 0.0027,$$

$$e_4 < \frac{0.0027^2}{2.8} = \frac{7.29 \times 10^{-6}}{2.8} < 2.7 \times 10^{-6},$$

$$e_5 < \frac{(2.7 \times 10^{-6})^2}{2.8} = \frac{7.29 \times 10^{-12}}{2.8} < 2.7 \times 10^{-12} < 10^{-11}.$$

And $e_5 > 0$ by (a). ■

↪ *The digits of accuracy go roughly* $1, 2, 5, 11$*: doubling each time. That's* ***quadratic convergence****.*

Answer. Proofs as above.

Common errors.

- Claiming $x_n > \sqrt{2}$ for $n = 1$ as well ($x_1 = 1$).
- In (b), using a calculator value of x_5, which the question forbids.

How the marks are earned.

M1 **(a)** All $x_n > 0$ (induction)
A1 $x_{n+1} - \sqrt{2} > 0$, strict since x_n is rational and $\neq \sqrt{2}$
M1 $x_{n+1} - x_n = \dfrac{2 - x_n^2}{2x_n}$
A1 < 0 for $n \geq 2$
M1 **(b)** $e_{n+1} = \dfrac{e_n^2}{2x_n} < \dfrac{e_n^2}{2\sqrt{2}} < \dfrac{e_n^2}{2.8}$
A1 Correct justification ($x_n > \sqrt{2}$, $2\sqrt{2} > 2.8$)
M1 $e_2 < 0.086$
A1 $e_3 < 0.0027$, $e_4 < 2.7 \times 10^{-6}$
A1 $e_5 < 2.7 \times 10^{-12} < 10^{-11}$

■

Question 12 *Iteration: Pell's equation* 4 marks

> *The sequence* $x_1, x_2, \ldots$ *is defined by* $x_1 = 1$ *and* $x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$.
>
> *Write* $x_n = \dfrac{p_n}{q_n}$, *where* $p_1 = q_1 = 1$, $p_{n+1} = p_n^2 + 2q_n^2$ *and* $q_{n+1} = 2p_nq_n$. *Prove that* $p_n^2 - 2q_n^2 = 1$ *for all* $n \geq 2$.

Solution.

The idea.

First check that the recurrences for p_n *and* q_n *really produce the sequence. Then show that* $p^2 - 2q^2$ *is* ***squared*** *at each step, so once it equals 1 it stays 1.*

Step 1. The recurrence matches.

If $x_n = \frac{p_n}{q_n}$ then

$$x_{n+1} = \frac{1}{2}\left(\frac{p_n}{q_n} + \frac{2q_n}{p_n}\right) = \frac{p_n^2 + 2q_n^2}{2p_nq_n} = \frac{p_{n+1}}{q_{n+1}}.\checkmark$$

Step 2. The key algebra.

$$p_{n+1}^2 - 2q_{n+1}^2 = (p_n^2 + 2q_n^2)^2 - 2(2p_nq_n)^2 = p_n^4 + 4p_n^2q_n^2 + 4q_n^4 - 8p_n^2q_n^2 = (p_n^2 - 2q_n^2)^2.$$

Step 3. Induction.

$n = 2$: $p_2 = 3$, $q_2 = 2$, and $9 - 8 = 1$ ✓. If $p_n^2 - 2q_n^2 = 1$, then $p_{n+1}^2 - 2q_{n+1}^2 = 1^2 = 1$. So it holds for all $n \geq 2$. ■ (It fails at $n = 1$, where $1 - 2 = -1$, but squaring fixes the sign.)

↪ $p^2 - 2q^2 = 1$ *is* ***Pell's equation****. Dividing by* q^2*:* $\left(\frac{p}{q}\right)^2 = 2 + \frac{1}{q^2}$, *which shows directly hou close* $\frac{p}{q}$ *is to* $\sqrt{2}$.

Answer. Proof as above.

How the marks are earned.

M1 Checks $x_{n+1} = \dfrac{p_n^2 + 2q_n^2}{2p_nq_n}$ matches the recurrence
M1 $p_{n+1}^2 - 2q_{n+1}^2 = (p_n^2 + 2q_n^2)^2 - 8p_n^2q_n^2$
A1 $= (p_n^2 - 2q_n^2)^2$
A1 $n = 2$: $9 - 8 = 1$; induction completes

■

Question 13 *Tangents to cubics* 7 marks

> *The point P has coordinates (p, p^3), where $p \neq 0$, and lies on the curve C with equation $y = x^3$.*
>
> ***(a)*** *Show that the tangent to C at P has equation $y = 3p^2x - 2p^3$.* *(3)*
>
> ***(b)*** *Show that this tangent meets C again at the point Q with x-coordinate $-2p$.* *(4)*

Solution.

The idea.

*A tangent touches the curve at P, so "curve minus tangent" has a **double root** at $x = p$. That makes the third intersection easy, and the area a clean integral. Watch the video to see the chain of tangents zig-zag outwards while the areas grow by a factor of 16 each time.*

Step 1. (a) The tangent at P.

$\frac{dy}{dx} = 3x^2$, so the gradient at P is $3p^2$:

$$y - p^3 = 3p^2(x - p) \quad \Longrightarrow \quad y = 3p^2x - 3p^3 + p^3 = 3p^2x - 2p^3. \quad \blacksquare$$

Step 2. (b) Intersect the tangent with the curve.

$$x^3 = 3p^2x - 2p^3 \quad \Longrightarrow \quad x^3 - 3p^2x + 2p^3 = 0.$$

Step 3. (b) Use the double root.

Because the line is tangent at $x = p$, this cubic has $x = p$ as a **repeated** root, so $(x - p)^2 = x^2 - 2px + p^2$ is a factor. The remaining factor must be $(x + 2p)$ to match the x^3 and constant terms ($p^2 \times 2p = 2p^3$):

$$x^3 - 3p^2x + 2p^3 = (x - p)^2(x + 2p).$$

Check the x term: $-2p \cdot 2p\,x + p^2x = -3p^2x$ ✓. So Q has $x = -2p$. $\blacksquare$

↪ *Quicker still: the roots of $x^3 + 0x^2 - 3p^2x + 2p^3$ add to 0, so $p + p + x_Q = 0$.*

Answer. Proofs as above.

How the marks are earned.

M1 (a) $\dfrac{dy}{dx} = 3x^2$, gradient $3p^2$
M1 $y - p^3 = 3p^2(x - p)$
A1* Simplifies to the result
M1 (b) $x^3 - 3p^2x + 2p^3 = 0$
M1 Notes $x = p$ is a repeated root (tangent)
A1 $(x - p)^2(x + 2p) = 0$
A1* $x = -2p$

Question 14 *Tangents to cubics: area* 7 marks

> *The tangent to the curve $y = x^3$ at the point $P(p, p^3)$, where $p \neq 0$, has equation $y = 3p^2x - 2p^3$, and it meets the curve again at the point with x-coordinate $-2p$.*
>
> *Show that the area of the region enclosed between C and the tangent at P is $\dfrac{27p^4}{4}$.*

Solution.

The idea.

"Curve minus tangent" has a double root at $x = p$ and a single root at $x = -2p$. Substituting $u = x - p$ makes the integral of $(x-p)^2(x+2p)$ simple.

Step 1. Set up the area.

Between $x = -2p$ and $x = p$ the region lies between the curve and the tangent, so the area is $\left|\int (\text{curve} - \text{tangent})\,dx\right|$:

$$\text{Area} = \left| \int_{-2p}^{p} (x-p)^2(x+2p)\,dx \right|.$$

Step 2. A good substitution.

Let $u = x - p$. Then $x + 2p = u + 3p$, and the limits become $u = -3p$ and $u = 0$:

$$\int_{-3p}^{0} u^2(u+3p)\,du = \int_{-3p}^{0} \left(u^3 + 3pu^2\right) du = \left[\frac{u^4}{4} + pu^3\right]_{-3p}^{0}.$$

Step 3. Evaluate.

$$0 - \left(\frac{81p^4}{4} + p(-27p^3)\right) = -\frac{81p^4}{4} + 27p^4 = \frac{108p^4 - 81p^4}{4} = \frac{27p^4}{4}.$$

This is positive whatever the sign of p (it is a fourth power), and the substitution worked for either sign, so

$$\text{Area} = \frac{27p^4}{4}. \quad \blacksquare$$

Answer. $\dfrac{27p^4}{4}$

Common errors.

- Integrating "tangent minus curve" and getting a negative area without taking the modulus.
- Assuming $p > 0$ without comment.

How the marks are earned.

M1 Area $= \left|\int_{-2p}^{p} \left(x^3 - 3p^2x + 2p^3\right) dx\right|$

B1 Integrand $= (x-p)^2(x+2p)$

M1 Substitution $u = x - p$ (or direct integration)

A1 $\int_{-3p}^{0} u^2(u+3p)\, du$

A1 $\left[\frac{u^4}{4} + pu^3\right]$

M1 Evaluates: $-\frac{81p^4}{4} + 27p^4$

A1* $\dfrac{27p^4}{4}$, valid for either sign of p

∎

Question 15 *Tangents to cubics: a geometric sequence* 6 marks

For $p \neq 0$, the tangent to $y = x^3$ at $P(p, p^3)$ meets the curve again at the point with x-coordinate $-2p$, and the area enclosed between the curve and this tangent is $\dfrac{27p^4}{4}$. The tangent at Q (the second point) meets the curve again at R, the tangent at R meets it again at S, and so on.

(a) *Show that the areas enclosed between C and successive tangents form a geometric sequence, and state its common ratio.* *(3)*

(b) *Show that the same common ratio arises for **every** cubic curve $y = ax^3 + bx^2 + cx + d$, where $a \neq 0$.* *(3)*

A video explanation of this question is at lemmamathematics.com/papers/pure-5#q15.

Solution.

The idea.

Each new tangent point is -2 times the previous one, so lengths double, and the areas, which scale like the fourth power of length, multiply by 16. For a general cubic, measure distances from the point of inflection. The video shows the chain of tangents.

Step 1. (a) Follow the chain.

The tangent at a point with x-coordinate t meets the curve again at $-2t$. So the tangent points go

$$p,\ -2p,\ 4p,\ -8p,\ \ldots$$

and the areas are

$$\frac{27p^4}{4},\ \frac{27(2p)^4}{4},\ \frac{27(4p)^4}{4},\ \ldots = \frac{27p^4}{4} \times 1,\ 16,\ 256,\ \ldots$$

A geometric sequence with common ratio $2^4 = 16$.

Step 2. (b) Curve minus tangent in general.

For $y = ax^3 + bx^2 + cx + d$, take the tangent at $x = t$. The difference (curve minus tangent) is a cubic with leading coefficient a and a double root at t:

$$a(x - t)^2(x - s),$$

where s is the other intersection. The sum of the roots of $ax^3 + bx^2 + \ldots$ is $-\frac{b}{a}$ (subtracting a line doesn't change the x^3 or x^2 terms), so

$$2t + s = -\frac{b}{a}.$$

Step 3. (b) Measure from the point of inflection.

Let $k = -\frac{b}{3a}$ (the x-coordinate of the point of inflection). Then $s = 3k - 2t$, so

$$s - k = -2(t - k)\,:$$

each new tangent point is twice as far from k, on the other side.

Step 4. (b) The area.

With $L = s - t = -3(t - k)$, the same substitution as for $y = x^3$ gives

$$\left| \int_t^s a(x-t)^2(x-s)\,dx \right| = \frac{|a|\,L^4}{12} = \frac{|a| \cdot 81(t-k)^4}{12}.$$

Doubling $|t - k|$ multiplies this by $2^4 = 16$. So the ratio is **16 for every cubic**. ■

Check with $y = x^3$: $a = 1$, $k = 0$, giving $\frac{81p^4}{12} = \frac{27p^4}{4}$ ✓.

Answer. (a) common ratio **16**

Common errors.

- Forgetting that each new point is on the **other** side of the origin (or point of inflection).
- Assuming the general cubic has no x^2 term without translating it first.

Examiner's note.

The single most useful trick in this question: when a line is tangent to a polynomial, the difference has a **repeated** root, and the sum of the roots does the rest.

How the marks are earned.

M1 **(a)** The next tangent point has $x = -2p$, then $4p$, $-8p$, …

A1 Next area $= \frac{27(2p)^4}{4} = 16 \times \frac{27p^4}{4}$

A1 Geometric with common ratio 16

M1 **(b)** Curve minus tangent at $x = t$ is $a(x-t)^2(x-s)$ with $2t + s = -\frac{b}{a}$

M1 Area $= \dfrac{|a|\,|s-t|^4}{12}$ and $s - k = -2(t-k)$ where $k = -\frac{b}{3a}$

A1 Distances from k double, so areas scale by $2^4 = 16$

■