LEMMA PRACTICE PAPER · MARK SCHEME

ULTRA

Pure Mathematics

Ultra paper

GENERAL MARKING GUIDANCE

- All candidates must receive the same treatment: mark positively, and give credit for what is shown rather than penalising what is missing.
- **M** marks: method marks, for knowing and applying a correct method to the problem, even if there are small slips.
- **A** marks: accuracy marks, which can only be awarded if the relevant M mark has been earned.
- **B** marks: unconditional accuracy marks, independent of method.
- **dM**: a method mark that depends on the previous M mark. **ft**: follow through from an earlier error. *****: the answer is given in the question, so full working must be seen with no errors. **cao**: correct answer only.
- Correct alternative methods earn full credit. The model solutions show several of them.

QUESTION	SCHEME	MARKS	NOTES
1 (a)	Squares: $x^2 = 5 + 2\sqrt{6}$	**M1**	
	$(x^2 - 5)^2 = 24$	**A1**	
	$x^4 - 10x^2 + 1 = 0$	**A1***	
(b)	Assumes $\frac{p}{q}$ (lowest terms) is a root and multiplies by q^4: $p^4 - 10p^2q^2 + q^4 = 0$	**M1**	
	$q \mid p^4$, and $\gcd(p, q) = 1$, so $q = \pm 1$	**A1**	
	$p \mid q^4$, so $p = \pm 1$	**A1**	
	Tests $x = \pm 1$: $1 - 10 + 1 = -8 \neq 0$	**B1**	
	No rational roots, so $\sqrt{2} + \sqrt{3}$ is irrational	**A1**	
(c)	$\sqrt{3} - \sqrt{2} = \dfrac{1}{\sqrt{3} + \sqrt{2}}$	**M1**	
	The reciprocal of an irrational number is irrational (if $\frac{1}{x} = \frac{a}{b}$ then $x = \frac{b}{a}$)	**A1**	
	Total for Question 1	**(10)**	

QUESTION	SCHEME	MARKS	NOTES
2 (a)	Expands $(1 - x)^4$ and multiplies by x^4	**M1**	
	$x^8 - 4x^7 + 6x^6 - 4x^5 + x^4$	**A1**	
	Expands $(1 + x^2)(x^6 - 4x^5 + 5x^4 - 4x^2 + 4)$	**M1**	

QUESTION	SCHEME	MARKS	NOTES
	Obtains $x^8 - 4x^7 + 6x^6 - 4x^5 + x^4 + 4$ and completes	**A1***	
(b)	Splits the integrand: polynomial $-\dfrac{4}{1+x^2}$	**M1**	
	Integrates the polynomial	**M1**	
	Obtains $\frac{22}{7}$	**A1**	
	$\int_0^1 \dfrac{4}{1+x^2}\,dx = 4\arctan 1 = \pi$	**M1**	
	Completes to $\frac{22}{7} - \pi$	**A1***	
(c)	Integrand positive on $(0,1)$, so $\pi < \frac{22}{7}$	**B1**	
	Total for Question 2	**(10)**	

QUESTION	SCHEME	MARKS	NOTES
3	$\frac{1}{2} \le \dfrac{1}{1+x^2} \le 1$ on $[0,1]$	**M1**	
	Bounds the integral by $\frac{1}{2}J$ and J, where $J = \int_0^1 x^4(1-x)^4\,dx$	**M1**	
	$x^4(1-x)^4 = x^8 - 4x^7 + 6x^6 - 4x^5 + x^4$	**B1**	
	Integrates term by term	**M1**	
	$J = \frac{1}{630}$	**A1**	
	Strict inequalities justified (bounds attained only at endpoints)	**A1**	
	$\frac{1}{1260} < \frac{22}{7} - \pi < \frac{1}{630}$	**A1***	
	Rearranges for π	**M1**	
	$3.14127 < \pi < 3.14207$	**A1**	
	$\pi = 3.14$ to 2 d.p., with reason (both bounds round to 3.14)	**A1**	
	Total for Question 3	**(10)**	

QUESTION	SCHEME	MARKS	NOTES
4 (a)	$(4a+1)(4b+1) = 16ab + 4a + 4b + 1$	**M1**	
	$= 4(4ab + a + b) + 1$	**A1***	
(b)	An odd number has only odd prime factors, each of the form $4k+1$ or $4k+3$	**M1**	
	If all were $4k+1$, the product would be $4k+1$ (repeated use of (a))	**M1**	
	Contradiction, so at least one prime factor is $4k+3$	**A1**	
(c)	$N = 4p_1 \cdots p_m - 1$ is of the form $4k+3$ and $N > 1$	**B1**	

QUESTION	SCHEME	MARKS	NOTES
	By (b), N has a prime factor q of the form $4k+3$	**M1**	
	So $q = p_i$ for some i	**A1**	
	p_i divides $4p_1 \cdots p_m$ and N, so $p_i \mid 1$	**M1**	
	Contradiction: infinitely many such primes	**A1**	
	Total for Question 4	**(10)**	

QUESTION	SCHEME	MARKS	NOTES
5 (i)	$e - S_n = \sum_{k=n+1}^{\infty} \frac{1}{k!} > 0$	**B1**	
	Factors out $\frac{1}{(n+1)!}$	**M1**	
	$1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \cdots$	**A1**	
	Compares with the geometric series in $\frac{1}{n+1}$	**M1**	
	Strict inequality justified	**A1**	
	Sums: $\frac{n+1}{n}$	**M1**	
	$\frac{1}{n!\,n}$	**A1***	
(ii)	$S_2 = 2.5$	**B1**	
	Uses $n = 2$ in (i)	**M1**	
	$2.5 < e < 2.75$	**A1**	
	Total for Question 5	**(10)**	

QUESTION	SCHEME	MARKS	NOTES
6 (a)	Assumes $e = \frac{p}{q}$ with p, q positive integers	**B1**	
	$q!\,e = p\,(q-1)!$ is an integer	**M1**	
	$\frac{q!}{k!}$ is an integer for $0 \le k \le q$	**M1**	
	So N is an integer	**A1**	
	Uses the bound with $n = q$	**M1**	
	$0 < N < \frac{1}{q} \le 1$	**A1**	
	No integer strictly between 0 and 1, so e is irrational	**A1**	
(b)	$n!\,n > 10^{10}$	**M1**	
	$12! \times 12 \approx 5.7 \times 10^9$ fails, $13! \times 13 \approx 8.1 \times 10^{10}$ works: $n = 13$	**A1**	
	$e - S_{12} > \frac{1}{13!} \approx 1.6 \times 10^{-10}$, so 12 really is not enough	**B1**	

QUESTION	SCHEME	MARKS	NOTES
	Total for Question 6	**(10)**	

QUESTION	SCHEME	MARKS	NOTES
7 (a)(i)	$x = y = 0$: $f(0) = 2f(0)$	**M1**	
	$f(0) = 0$	**A1**	
(a)(ii)	Expands $g(x+y) = f(x+y) - (x+y)^2$	**M1**	
	$= g(x) + g(y)$	**A1***	
(b)	Differentiates with respect to y: $f'(x+y) = f'(y) + 2x$	**M1**	
	$y = 0$: $f'(x) = 2x + c$	**A1**	
	Integrates	**M1**	
	$f(x) = x^2 + cx + d$	**A1**	
	$f(0) = 0 \Rightarrow d = 0$	**M1**	
	$f(x) = x^2 + cx$, verified	**A1**	
	Total for Question 7	**(10)**	

QUESTION	SCHEME	MARKS	NOTES
8	$y = 0$: $h(x) = h(x)h(0)$	**M1**	
	Uses a point where $h \neq 0$: $h(0) = 1$	**A1**	
	$h(x)h(-x) = 1$, so h is never 0	**M1**	
	$h(x) = h\left(\frac{x}{2}\right)^2 \geq 0$	**M1**	
	Hence $h(x) > 0$	**A1**	
	Differentiates with respect to y at $y = 0$: $h'(x) = kh(x)$, $k = h'(0)$	**M1**	
	Solves rigorously, e.g. $\frac{d}{dx}\left(h(x)e^{-kx}\right) = 0$	**M1**	
	$h(x) = Ce^{kx}$	**A1**	
	$C = h(0) = 1$	**A1**	
	$h(x) = e^{kx}$ for any real k, verified	**A1**	
	Total for Question 8	**(10)**	

QUESTION	SCHEME	MARKS	NOTES
9 (a)	$a - b(\pi - x) = bx$, using $a = b\pi$	**M1**	
	$f(\pi - x) = f(x)$	**A1**	
	Differentiates j times: $(-1)^j f^{(j)}(\pi - x) = f^{(j)}(x)$	**M1**	
	Puts $x = 0$	**A1**	

QUESTION	SCHEME	MARKS	NOTES
(b)	$f(x) = \dfrac{1}{n!}\sum_{m=n}^{2n} c_m x^m$ with integer c_m	**M1**	
	$f^{(j)}(0) = \dfrac{j!\,c_j}{n!}$	**M1**	
	$\dfrac{j!}{n!}$ is an integer for $j \geq n$	**B1**	
	Zero for $j < n$ and $j > 2n$; so always an integer	**A1**	
	Uses (a) for the values at π	**M1**	
	$F(0) + F(\pi)$ is an integer	**A1**	
	Total for Question 9	**(10)**	

QUESTION	SCHEME	MARKS	NOTES
10 (a)	$F'' = f'' - f^{(4)} + \cdots$, with $f^{(2n+2)} = 0$ (degree $2n$)	**M1**	
	$F'' + F = f$	**A1**	
	$\dfrac{d}{dx}\left(F' \sin x - F\cos x\right) = (F'' + F)\sin x$	**M1**	
	Evaluates between 0 and π: $F(\pi) + F(0)$	**A1***	
(b)	$f(x)\sin x > 0$ on $(0, \pi)$, so the integral is positive	**M1**	
	$x \leq \pi$ and $0 \leq a - bx \leq a$ on $[0, \pi]$	**M1**	
	Integral $< \dfrac{\pi^{n+1}a^n}{n!}$	**A1**	
	$\dfrac{c^n}{n!} \to 0$, with a reason (e.g. factors $\frac{c}{m} \leq \frac{1}{2}$ once $m \geq 2c$)	**M1**	
	Chooses n so that the bound is less than 1	**A1**	
	Integer strictly between 0 and 1: π is irrational	**A1**	
	Total for Question 10	**(10)**	

TOTAL FOR PAPER: 100 MARKS