Name _____ Date _____ Score _____ / 100

LEMMA PRACTICE PAPER

ULTRA

# Pure Mathematics

*Ultra paper*

Paper reference **LPU/P** Time **3 hours**

| TOTAL MARKS | 100 |
|---|---|

**You must have:** pen, pencil, ruler, graph paper and a calculator.

## INSTRUCTIONS

- Use black ink or ball-point pen; pencil may be used for diagrams.
- Answer **all** questions, in the spaces provided. If you need more space, continue on extra paper and label it clearly.
- Show all your working. An answer with no supporting method may get no marks.
- Give answers to 3 significant figures unless a question says otherwise, and give exact answers when asked.

## INFORMATION

- There are **10** questions, and the paper is out of **100** marks.
- The marks for each part of a question are shown in brackets, for example **(3)**. Use them to judge how much to write.
- The questions are in roughly increasing order of difficulty.
- Difficulty: **Ultra**. Famous theorems proved from scratch with school tools. Expect to spend real time on each question. Finishing one is an achievement; finishing all ten is exceptional.

## ADVICE

- Read each question carefully before you start.
- Keep an eye on the time: roughly one minute per mark.
- Check your answers if you have time at the end.
- The mark scheme, very detailed model solutions and video explanations are free at **lemmamathematics.com/papers/pure-ultra**.

**1** This question shows that $\sqrt{2}+\sqrt{3}$ is irrational.

**(a)** Show that $x=\sqrt{2}+\sqrt{3}$ satisfies $x^4-10x^2+1=0$. **(3)**

**(b)** By considering a rational root $\frac{p}{q}$ in its lowest terms, prove that the equation $x^4-10x^2+1=0$ has no rational roots. Deduce that $\sqrt{2}+\sqrt{3}$ is irrational. **(5)**

**(c)** Deduce that $\sqrt{3} - \sqrt{2}$ is irrational. **(2)**

**(Total for Question 1 is 10 marks)**

**2** This question shows that $\frac{22}{7}$ is larger than $\pi$, and by how much.

**(a)** Show that **(4)**

$$x^4(1-x)^4 \equiv (1+x^2)\left(x^6 - 4x^5 + 5x^4 - 4x^2 + 4\right) - 4.$$

**(b)** Hence show that **(5)**

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx = \frac{22}{7} - \pi.$$

**(c)** Deduce that $\pi < \frac{22}{7}$. **(1)**

**(Total for Question 2 is 10 marks)**

**3** You are given that **(10)**

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx = \frac{22}{7} - \pi.$$

By finding upper and lower bounds for $\frac{1}{1+x^2}$ on $[0,1]$, show that

$$\frac{1}{1260} < \frac{22}{7} - \pi < \frac{1}{630},$$

and deduce the value of $\pi$ correct to 2 decimal places.

**(Total for Question 3 is 10 marks)**

**4** This question proves that there are infinitely many primes of the form $4k + 3$.

**(a)** Show that the product of two integers of the form $4k + 1$ is also of the form $4k + 1$. **(2)**

**(b)** Deduce that every positive integer of the form $4k + 3$ has at least one prime factor of the form $4k + 3$. **(3)**

**(c)** Suppose that $p_1, p_2, \ldots, p_m$ are **all** the primes of the form $4k + 3$. By considering $N = 4p_1p_2 \cdots p_m - 1$, obtain a contradiction. **(5)**

**(Total for Question 4 is 10 marks)**

**5** You may assume that $e = \sum_{k=0}^{\infty} \frac{1}{k!}$. For each positive integer $n$, let $S_n = \sum_{k=0}^{n} \frac{1}{k!}$. **(10)**

(i) Show that

$$0 < e - S_n < \frac{1}{n!\, n}.$$

(ii) Deduce that $2.5 < e < 2.75$.

**(Total for Question 5 is 10 marks)**

**6** You may assume that $e = \sum_{k=0}^{\infty} \frac{1}{k!}$ and that, with $S_n = \sum_{k=0}^{n} \frac{1}{k!}$,

$$0 < e - S_n < \frac{1}{n!\,n} \qquad \textbf{for every positive integer } n.$$

**(a)** Suppose that $e = \dfrac{p}{q}$, where $p$ and $q$ are positive integers. By considering $N = q!\,(e - S_q)$, prove that $e$ is irrational. **(7)**

**(b)** Find the least value of $n$ for which the given bound guarantees that $S_n$ approximates $e$ to within $10^{-10}$, and show that, for this question, the bound is not wasteful: the previous value of $n$ really is not accurate enough. **(3)**

**(Total for Question 6 is 10 marks)**

**7** **(a)** The function $f : \mathbb{R} \to \mathbb{R}$ satisfies **(4)**

$$f(x + y) = f(x) + f(y) + 2xy \qquad \text{for all real } x, y.$$

(i) Show that $f(0) = 0$. (ii) Show that $g(x) = f(x) - x^2$ satisfies $g(x + y) = g(x) + g(y)$ for all real $x, y$.

**(b)** Given also that $f$ is differentiable, find all possible functions $f$, and verify that they satisfy the equation. **(6)**

**(Total for Question 7 is 10 marks)**

**8** Find all differentiable functions $h : \mathbb{R} \to \mathbb{R}$, not identically zero, such that **(10)**

$$h(x + y) = h(x)h(y) \qquad \text{for all real } x, y.$$

You should prove that $h(0) = 1$, that $h(x) > 0$ for all $x$, and that your answer is complete.

**(Total for Question 8 is 10 marks)**

**9** In this question you will prove that $\pi$ is irrational (Niven, 1947). Suppose, for contradiction, that $\pi = \dfrac{a}{b}$, where $a$ and $b$ are positive integers. For a positive integer $n$, define

$$f(x) = \frac{x^n(a - bx)^n}{n!}$$

and

$$F(x) = f(x) - f''(x) + f^{(4)}(x) - \cdots + (-1)^n f^{(2n)}(x).$$

**(a)** Show that $f(\pi - x) = f(x)$, and deduce that $f^{(j)}(\pi) = (-1)^j f^{(j)}(0)$ for every $j \geq 0$. **(4)**

**(b)** Show that $f^{(j)}(0)$ is an integer for every $j \geq 0$. Deduce that $F(0) + F(\pi)$ is an integer. **(6)**

**(Total for Question 9 is 10 marks)**

**10** As in the previous question, suppose that $\pi = \dfrac{a}{b}$, where $a$ and $b$ are positive integers, and for a positive integer $n$ let

$$f(x) = \frac{x^n(a-bx)^n}{n!}, \qquad F(x) = f(x) - f''(x) + f^{(4)}(x) - \cdots + (-1)^n f^{(2n)}(x).$$

You may assume that $F(0) + F(\pi)$ is an integer.

**(a)** Show that $F''(x) + F(x) = f(x)$, and hence that **(4)**

$$\int_0^\pi f(x)\sin x\,dx = F(\pi) + F(0).$$

**(b)** Show that **(6)**

$$0 < \int_0^\pi f(x)\sin x\,dx < \frac{\pi^{n+1}a^n}{n!},$$

and hence complete the proof that $\pi$ is irrational.

**(Total for Question 10 is 10 marks)**

**TOTAL FOR PAPER IS 100 MARKS**

END OF PAPER