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Pure Mathematics

Ultra paper: model solutions

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How to use these solutions. Try each question under timed conditions first, then mark it with the mark scheme. Read the full solution for every question, even the ones you got right: they often show a faster method, a check, or a common error to avoid. Each solution begins with *the idea*, then works through numbered steps; remarks marked ↪ explain *why*. Each ends with the boxed answer and how the marks are earned.

Question 1 *Irrational numbers* 10 marks

This question shows that $\sqrt{2}+\sqrt{3}$ is irrational.

(a) *Show that $x=\sqrt{2}+\sqrt{3}$ satisfies $x^4-10x^2+1=0$.* *(3)*

(b) *By considering a rational root $\frac{p}{q}$ in its lowest terms, prove that the equation $x^4-10x^2+1=0$ has no rational roots. Deduce that $\sqrt{2}+\sqrt{3}$ is irrational.* *(5)*

(c) *Deduce that $\sqrt{3}-\sqrt{2}$ is irrational.* *(2)*

Solution.

The idea.

*Square twice to get rid of the surds, giving a polynomial with **integer** coefficients. Then a neat divisibility argument shows that such a polynomial can only have rational roots of a very restricted kind, and none of them work.*

Step 1. (a) Square once.

$$x^2=(\sqrt{2}+\sqrt{3})^2=2+2\sqrt{2}\sqrt{3}+3=5+2\sqrt{6}.$$

↪ $\sqrt{2}\sqrt{3}=\sqrt{6}$, *and the cross term appears twice:* $(a+b)^2=a^2+2ab+b^2$.

Step 2. (a) Isolate the surd and square again.

$$x^2-5=2\sqrt{6} \quad\Longrightarrow\quad (x^2-5)^2=4\times 6=24.$$

Expand:

$$x^4-10x^2+25=24 \quad\Longrightarrow\quad x^4-10x^2+1=0. \quad \blacksquare$$

Step 3. (b) Suppose there is a rational root.

Let $x=\frac{p}{q}$ be a root, with p,q integers, $q\neq 0$, and the fraction in **lowest terms**: $\gcd(p,q)=1$. Then

$$\frac{p^4}{q^4}-\frac{10p^2}{q^2}+1=0.$$

Multiply by q^4:

$$p^4 - 10p^2q^2 + q^4 = 0.$$

Step 4. (b) What q must be.
Rearrange: $p^4 = 10p^2q^2 - q^4 = q^2(10p^2 - q^2)$. So q divides p^4. But p and q have no common prime factor, so no prime can divide both q and p^4. Hence q has no prime factors at all: $q = \pm 1$.

Step 5. (b) What p must be.
Similarly $q^4 = p^2(10q^2 - p^2)$, so p divides q^4, and the same argument gives $p = \pm 1$. So the only possible rational roots are $x = \pm 1$.

Step 6. (b) Test them.
$x = 1$ and $x = -1$ both give $1 - 10 + 1 = -8 \neq 0$. So the equation has **no** rational roots.

Step 7. (b) Conclude.
$\sqrt{2} + \sqrt{3}$ is a root (part (a)) but no root is rational. So $\sqrt{2} + \sqrt{3}$ is irrational. ■

↪ *This is the **rational root theorem**: for an integer polynomial, any rational root $\frac{p}{q}$ in lowest terms has p dividing the constant term and q dividing the leading coefficient.*

Step 8. (c) Rationalise.

$$(\sqrt{3} - \sqrt{2})(\sqrt{3} + \sqrt{2}) = 3 - 2 = 1 \quad \Longrightarrow \quad \sqrt{3} - \sqrt{2} = \frac{1}{\sqrt{3} + \sqrt{2}}.$$

Step 9. (c) Reciprocals of irrationals.
If $\sqrt{3} - \sqrt{2}$ were rational, say $\frac{a}{b}$ with $a \neq 0$, then $\sqrt{3} + \sqrt{2} = \frac{b}{a}$ would be rational too, contradicting (b). So $\sqrt{3} - \sqrt{2}$ is irrational. ■

Answer. (a), (b), (c): proofs as above.

Common errors.

- "$\sqrt{2}$ and $\sqrt{3}$ are irrational, so their sum is": false in general ($\sqrt{2} + (-\sqrt{2}) = 0$).
- Forgetting to say the fraction is in lowest terms, which the divisibility step needs.

How the marks are earned.
M1 **(a)** Squares: $x^2 = 5 + 2\sqrt{6}$
A1 $(x^2 - 5)^2 = 24$
A1* $x^4 - 10x^2 + 1 = 0$
M1 **(b)** Assumes $\frac{p}{q}$ (lowest terms) is a root and multiplies by q^4: $p^4 - 10p^2q^2 + q^4 = 0$
A1 $q \mid p^4$, and $\gcd(p, q) = 1$, so $q = \pm 1$
A1 $p \mid q^4$, so $p = \pm 1$
B1 Tests $x = \pm 1$: $1 - 10 + 1 = -8 \neq 0$
A1 No rational roots, so $\sqrt{2} + \sqrt{3}$ is irrational
M1 **(c)** $\sqrt{3} - \sqrt{2} = \dfrac{1}{\sqrt{3} + \sqrt{2}}$
A1 The reciprocal of an irrational number is irrational (if $\frac{1}{x} = \frac{a}{b}$ then $x = \frac{b}{a}$)

∎

Question 2 *Approximations to π* 10 marks

This question shows that $\dfrac{22}{7}$ is larger than π, and by how much.

(a) *Show that* *(4)*

$$x^4(1-x)^4 \equiv (1+x^2)\left(x^6 - 4x^5 + 5x^4 - 4x^2 + 4\right) - 4.$$

(b) *Hence show that* *(5)*

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx = \frac{22}{7} - \pi.$$

(c) *Deduce that $\pi < \dfrac{22}{7}$.* *(1)*

Solution.

The idea.

*A beautiful fact: a single integral of something obviously **positive** equals $\frac{22}{7} - \pi$. Polynomial division splits the integrand into a polynomial (which gives $\frac{22}{7}$) and a multiple of $\frac{1}{1+x^2}$ (which gives π, via* arctan *).*

Step 1. (a) Expand the left-hand side.

From Pascal's triangle, $(1-x)^4 = 1 - 4x + 6x^2 - 4x^3 + x^4$, so

$$x^4(1-x)^4 = x^4 - 4x^5 + 6x^6 - 4x^7 + x^8.$$

Step 2. (a) Expand the product on the right.

$$\begin{aligned}(1+x^2)(x^6 - 4x^5 + 5x^4 - 4x^2 + 4) = {}& (x^6 - 4x^5 + 5x^4 - 4x^2 + 4) \\ & + (x^8 - 4x^7 + 5x^6 - 4x^4 + 4x^2).\end{aligned}$$

Step 3. (a) Collect powers.

power	from the 1	from the x^2	total
x^8		1	1
x^7		-4	-4
x^6	1	5	6
x^5	-4		-4
x^4	5	-4	1
x^2	-4	4	0
x^0	4		4

So the product is $x^8 - 4x^7 + 6x^6 - 4x^5 + x^4 + 4$.

Step 4. (a) Subtract 4.

$$(1+x^2)(x^6 - 4x^5 + 5x^4 - 4x^2 + 4) - 4 = x^8 - 4x^7 + 6x^6 - 4x^5 + x^4 = x^4(1-x)^4. \quad \blacksquare$$

↪ *Where does the quotient come from? Polynomial long division of $x^8 - 4x^7 + \cdots$ by $x^2 + 1$. The question hands it to you, but it's worth knowing how you'd find it.*

Step 5. (b) Split the integrand.

Divide the identity by $1+x^2$:

$$\frac{x^4(1-x)^4}{1+x^2} = x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2}.$$

Step 6. (b) The polynomial part.

$$\int_0^1 (x^6 - 4x^5 + 5x^4 - 4x^2 + 4)\,dx = \left[\frac{x^7}{7} - \frac{2x^6}{3} + x^5 - \frac{4x^3}{3} + 4x\right]_0^1$$

$$= \frac{1}{7} - \frac{2}{3} + 1 - \frac{4}{3} + 4 = \frac{1}{7} + (5-2) = \frac{1}{7} + 3 = \frac{22}{7}.$$

Step 7. (b) The arctan part.

$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$, so

$$\int_0^1 \frac{4}{1+x^2}\,dx = 4\big[\arctan x\big]_0^1 = 4\cdot\frac{\pi}{4} = \pi.$$

Step 8. (b) Combine.

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx = \frac{22}{7} - \pi. \quad \blacksquare$$

Step 9. (c) The integrand is positive.

For $0 < x < 1$: $x^4 > 0$, $(1-x)^4 > 0$ and $1+x^2 > 0$. A continuous function that is positive on $(0,1)$ has positive integral. So

$$\frac{22}{7} - \pi > 0 \quad\Longrightarrow\quad \pi < \frac{22}{7}. \quad \blacksquare$$

Answer. (c) $\pi < \frac{22}{7}$

Common errors.

- Arithmetic in (b): add the fractions carefully over a common denominator.
- Forgetting that $\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$.

How the marks are earned.

M1 **(a)** Expands $(1-x)^4$ and multiplies by x^4
A1 $x^8 - 4x^7 + 6x^6 - 4x^5 + x^4$
M1 Expands $(1+x^2)(x^6 - 4x^5 + 5x^4 - 4x^2 + 4)$
A1* Obtains $x^8 - 4x^7 + 6x^6 - 4x^5 + x^4 + 4$ and completes
M1 **(b)** Splits the integrand: polynomial $-\dfrac{4}{1+x^2}$
M1 Integrates the polynomial
A1 Obtains $\frac{22}{7}$
M1 $\int_0^1 \frac{4}{1+x^2}\,dx = 4\arctan 1 = \pi$
A1* Completes to $\frac{22}{7} - \pi$
B1 **(c)** Integrand positive on $(0,1)$, so $\pi < \frac{22}{7}$

∎

Question 3 *Bounding π* 10 marks

You are given that

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx = \frac{22}{7} - \pi.$$

By finding upper and lower bounds for $\dfrac{1}{1+x^2}$ *on* $[0,1]$*, show that*

$$\frac{1}{1260} < \frac{22}{7} - \pi < \frac{1}{630},$$

and deduce the value of π *correct to 2 decimal places.*

Solution.

The idea.

The integral is squeezed between $\frac{1}{2}J$ *and* J*, where* J *is the same integral without the denominator.* J *is just a polynomial integral, and the squeeze pins down* π *to within about* 0.0008*.*

Step 1. Bound the denominator.

On $[0,1]$, $1 \le 1+x^2 \le 2$, so

$$\frac{1}{2} \le \frac{1}{1+x^2} \le 1,$$

with equality only at the endpoints.

Step 2. The integral without the denominator.

$$J = \int_0^1 x^4(1-x)^4\,dx = \int_0^1 \left(x^8 - 4x^7 + 6x^6 - 4x^5 + x^4\right) dx = \frac{1}{9} - \frac{1}{2} + \frac{6}{7} - \frac{2}{3} + \frac{1}{5}.$$

Over the common denominator 630:

$$\frac{70 - 315 + 540 - 420 + 126}{630} = \frac{1}{630}.$$

Step 3. Squeeze.

Multiplying the positive function $x^4(1-x)^4$ by the bounds and integrating (strictly, since the bounds are only attained at endpoints):

$$\frac{1}{2}\cdot\frac{1}{630} < \frac{22}{7} - \pi < \frac{1}{630} \quad\Longrightarrow\quad \frac{1}{1260} < \frac{22}{7} - \pi < \frac{1}{630}. \qquad \blacksquare$$

Step 4. Pin down π.

Rearrange:

$$\frac{22}{7} - \frac{1}{630} < \pi < \frac{22}{7} - \frac{1}{1260}.$$

$\frac{22}{7} = 3.142857\ldots$, $\frac{1}{630} = 0.001587\ldots$, $\frac{1}{1260} = 0.000794\ldots$, so

$$3.14127 < \pi < 3.14207.$$

Every number in this interval rounds to 3.14, so $\pi = 3.14$ to 2 decimal places.

Answer. $3.14127 < \pi < 3.14207$, so $\pi = 3.14$ (2 d.p.)

Common errors.

- Using $\frac{1}{1+x^2} \geq 0$ as the lower bound, which gives nothing useful.
- Stating $\pi = 3.14$ without checking that **both** bounds round to it.

How the marks are earned.

M1 $\frac{1}{2} \leq \dfrac{1}{1+x^2} \leq 1$ on $[0,1]$

M1 Bounds the integral by $\frac{1}{2}J$ and J, where $J = \int_0^1 x^4(1-x)^4\,dx$

B1 $x^4(1-x)^4 = x^8 - 4x^7 + 6x^6 - 4x^5 + x^4$

M1 Integrates term by term

A1 $J = \frac{1}{630}$

A1 Strict inequalities justified (bounds attained only at endpoints)

A1* $\frac{1}{1260} < \frac{22}{7} - \pi < \frac{1}{630}$

M1 Rearranges for π

A1 $3.14127 < \pi < 3.14207$

A1 $\pi = 3.14$ to 2 d.p., with reason (both bounds round to 3.14)

∎

Question 4 *Primes* 10 marks

This question proves that there are infinitely many primes of the form $4k+3$.

(a) *Show that the product of two integers of the form $4k+1$ is also of the form $4k+1$.* *(2)*

(b) *Deduce that every positive integer of the form $4k+3$ has at least one prime factor of the form $4k+3$.* *(3)*

(c) *Suppose that $p_1, p_2, \ldots, p_m$ are **all** the primes of the form $4k+3$. By considering $N = 4p_1p_2\cdots p_m - 1$, obtain a contradiction.* *(5)*

Solution.

The idea.

Euclid's proof that there are infinitely many primes, with a twist. Multiply all the supposed primes together, adjust to get a number of the right form, and show it must have a prime factor that isn't on the list.

Step 1. (a) Multiply out.

$$(4a+1)(4b+1) = 16ab + 4a + 4b + 1 = 4(4ab + a + b) + 1,$$

which is of the form $4k+1$ with $k = 4ab + a + b$. ■

Step 2. (b) Look at the prime factors.

Let $n = 4k+3$. It is odd, so all its prime factors are odd. Every odd number leaves remainder 1 or 3 on division by 4, so each prime factor is of the form $4k+1$ or $4k+3$.

Step 3. (b) Suppose none is 4k + 3.

If every prime factor were of the form $4k+1$, then multiplying them together one at a time, using (a) at each step, the product n would be of the form $4k+1$. But n leaves remainder 3. Contradiction.

Step 4. (b) Conclude.

So at least one prime factor of n is of the form $4k+3$. ■

↪ *The same trick fails for $4k+1$: products of $4k+3$ numbers can be $4k+1$ (e.g. $3 \times 3 = 9$). That's why this half of the theorem is the easy one.*

Step 5. (c) The number N.

$$N = 4p_1p_2\cdots p_m - 1 = 4(p_1\cdots p_m - 1) + 3,$$

so N is of the form $4k+3$, and $N \geq 4 \times 3 - 1 = 11 > 1$ (since 3 is one of the p_i).

Step 6. (c) N has a prime factor on the list.

By (b), N has a prime factor q of the form $4k+3$. The list contains **all** such primes, so $q = p_i$ for some i.

Step 7. (c) The contradiction.

p_i divides $4p_1\cdots p_m$, and it divides N. So it divides their difference:

$$4p_1\cdots p_m - N = 1.$$

No prime divides 1. Contradiction. So there are infinitely many primes of the form $4k+3$. ■

Answer. Proof as above.

Common errors.

- Using $N = p_1 \cdots p_m + 1$, which need not be of the form $4k + 3$.
- Forgetting to check that $N > 1$ (so that it has a prime factor at all).

How the marks are earned.

M1 **(a)** $(4a+1)(4b+1) = 16ab + 4a + 4b + 1$
A1* $= 4(4ab + a + b) + 1$
M1 **(b)** An odd number has only odd prime factors, each of the form $4k + 1$ or $4k + 3$
M1 If all were $4k + 1$, the product would be $4k + 1$ (repeated use of (a))
A1 Contradiction, so at least one prime factor is $4k + 3$
B1 **(c)** $N = 4p_1 \cdots p_m - 1$ is of the form $4k + 3$ and $N > 1$
M1 By (b), N has a prime factor q of the form $4k + 3$
A1 So $q = p_i$ for some i
M1 p_i divides $4p_1 \cdots p_m$ and N, so $p_i \mid 1$
A1 Contradiction: infinitely many such primes

∎

Question 5 *The series for e* 10 marks

You may assume that $e = \displaystyle\sum_{k=0}^{\infty} \frac{1}{k!}$. *For each positive integer* n, *let* $S_n = \displaystyle\sum_{k=0}^{n} \frac{1}{k!}$.

(i) Show that

$$0 < e - S_n < \frac{1}{n!\,n}.$$

(ii) Deduce that $2.5 < e < 2.75$.

Solution.

The idea.

The tail of the series after n *terms starts with* $\frac{1}{(n+1)!}$, *and each later term is at least* $(n+1)$ *times smaller than the one before. So the tail is smaller than a geometric series, which you can sum.*

Step 1. (i) The tail.

$$e - S_n = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \cdots > 0.$$

Step 2. (i) Factor out the first term.

$$e - S_n = \frac{1}{(n+1)!}\left(1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \cdots\right).$$

Step 3. (i) Compare with a geometric series.

Each factor $n+2, n+3, \ldots$ is bigger than $n+1$, so (strictly, from the second term on)

$$1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \cdots < 1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots = \frac{1}{1 - \frac{1}{n+1}} = \frac{n+1}{n}.$$

Step 4. (i) Combine.

$$0 < e - S_n < \frac{1}{(n+1)!} \cdot \frac{n+1}{n} = \frac{1}{n!\,n}. \quad \blacksquare$$

Step 5. (ii) Use n equal to 2.

$S_2 = 1 + 1 + \frac{1}{2} = 2.5$, and the bound is $\frac{1}{2! \times 2} = \frac{1}{4}$:

$$2.5 < e < 2.75.$$

Answer. (ii) $2.5 < e < 2.75$

Common errors.

- Claiming $\leq$ instead of $<$ without noticing that the comparison is strict from the second term on.

How the marks are earned.

B1 (i) $e - S_n = \sum_{k=n+1}^{\infty} \dfrac{1}{k!} > 0$

M1 Factors out $\dfrac{1}{(n+1)!}$

A1 $1 + \dfrac{1}{n+2} + \dfrac{1}{(n+2)(n+3)} + \cdots$

M1 Compares with the geometric series in $\frac{1}{n+1}$

A1 Strict inequality justified

M1 Sums: $\dfrac{n+1}{n}$

A1* $\dfrac{1}{n!\,n}$

B1 (ii) $S_2 = 2.5$

M1 Uses $n = 2$ in (i)

A1 $2.5 < e < 2.75$

∎

Question 6 *The irrationality of e* 10 marks

You may assume that $e = \displaystyle\sum_{k=0}^{\infty} \frac{1}{k!}$ *and that, with* $S_n = \displaystyle\sum_{k=0}^{n} \frac{1}{k!}$,

$$0 < e - S_n < \frac{1}{n!\,n} \qquad \text{for every positive integer } n.$$

(a) *Suppose that* $e = \dfrac{p}{q}$, *where* p *and* q *are positive integers. By considering* $N = q!\,(e - S_q)$, *prove that* e *is irrational.* (7)

(b) *Find the least value of* n *for which the given bound guarantees that* S_n *approximates* e *to within* 10^{-10}, *and show that, for this question, the bound is not wasteful: the previous value of* n *really is not accurate enough.* (3)

Solution.

The idea.

If e *were* $\frac{p}{q}$, *then* $q!$ *times the tail of the series would be a whole number. But the given bound squeezes that number strictly between 0 and 1, where no whole number lives.*

Step 1. (a) Assume the opposite.

Suppose $e = \frac{p}{q}$ with p, q positive integers. Consider

$$N = q!\,(e - S_q) = q!\,e - \sum_{k=0}^{q} \frac{q!}{k!}.$$

Step 2. (a) N is an integer.

- $q!\,e = q! \cdot \frac{p}{q} = p\,(q-1)!$, an integer.
- For $0 \le k \le q$, $\frac{q!}{k!} = q(q-1)\cdots(k+1)$ is a product of integers (or 1 when $k = q$), so an integer.

A difference of integers is an integer, so $N \in \mathbb{Z}$.

Step 3. (a) N is strictly between 0 and 1.

By the given bound with $n = q$:

$$0 < e - S_q < \frac{1}{q!\,q} \quad \Longrightarrow \quad 0 < N < \frac{1}{q} \le 1.$$

Step 4. (a) Contradiction.

There is no integer strictly between 0 and 1. So the assumption is false, and e is irrational. ■

↪ *The proof works because the series converges faster than any* $\frac{1}{q!}$*-sized "grid" of fractions with denominator* q *could keep up with.*

Step 5. (b) The condition.

$$\frac{1}{n!\,n} < 10^{-10} \iff n!\,n > 10^{10}.$$

Step 6. (b) Try values.

- $n = 12$: $12! = 479\,001\,600$, times 12 is $5\,748\,019\,200 \approx 5.7 \times 10^9$. Too small.
- $n = 13$: $13! = 6\,227\,020\,800$, times 13 is $80\,951\,270\,400 \approx 8.1 \times 10^{10} > 10^{10}$ ✓.

So $n = 13$ (fourteen terms, $k = 0$ to 13).

Step 7. (b) The bound is sharp here.

For $n = 12$, the tail contains the single term $\frac{1}{13!}$, so

$$e - S_{12} > \frac{1}{13!} = 1.6 \times 10^{-10} > 10^{-10}.$$

So $n = 12$ genuinely fails: the bound didn't cost us anything.

Answer. (b) $n = 13$

Common errors.

- Forgetting that $q!\,e$ is an integer only *because* of the assumption.
- Showing $N < 1$ but not $N > 0$: both sides of the squeeze are needed.

How the marks are earned.

B1 (a) Assumes $e = \frac{p}{q}$ with p, q positive integers

M1 $q!\,e = p\,(q-1)!$ is an integer

M1 $\dfrac{q!}{k!}$ is an integer for $0 \le k \le q$

A1 So N is an integer

M1 Uses the bound with $n = q$

A1 $0 < N < \dfrac{1}{q} \le 1$

A1 No integer strictly between 0 and 1, so e is irrational

M1 (b) $n!\,n > 10^{10}$

A1 $12! \times 12 \approx 5.7 \times 10^{9}$ fails, $13! \times 13 \approx 8.1 \times 10^{10}$ works: $n = 13$

B1 $e - S_{12} > \dfrac{1}{13!} \approx 1.6 \times 10^{-10}$, so 12 really is not enough

∎

Question 7 *Functional equations* 10 marks

(a) *The function $f : \mathbb{R} \to \mathbb{R}$ satisfies* ***(4)***

$$f(x+y) = f(x) + f(y) + 2xy \qquad \text{for all real } x, y.$$

(i) Show that $f(0) = 0$. (ii) Show that $g(x) = f(x) - x^2$ satisfies $g(x+y) = g(x) + g(y)$ for all real x, y.

(b) *Given also that f is differentiable, find all possible functions f, and verify that they satisfy the equation.* ***(6)***

Solution.

The idea.

*A functional equation tells you how f behaves on **all** pairs of inputs. Two standard moves: substitute special values (like 0), and if f is differentiable, differentiate with respect to one variable while holding the other fixed. That turns the equation into a differential equation.*

Step 1. (a)(i) Put both variables to 0.

$$f(0+0) = f(0) + f(0) + 0 \quad \Longrightarrow \quad f(0) = 2f(0) \quad \Longrightarrow \quad f(0) = 0.$$

Step 2. (a)(ii) Expand.

$$g(x+y) = f(x+y) - (x+y)^2 = f(x) + f(y) + 2xy - x^2 - 2xy - y^2 = \big(f(x) - x^2\big) + \big(f(y) - y^2\big).$$

So $g(x+y) = g(x) + g(y)$. ■

↪ *Subtracting x^2 "absorbs" the $2xy$ term. That's the main idea of the question: f is a square plus an additive function.*

Step 3. (b) Differentiate with respect to y.

Fix x and differentiate both sides with respect to y (the chain rule gives $\frac{d}{dy} f(x+y) = f'(x+y)$):

$$f'(x+y) = f'(y) + 2x.$$

Step 4. (b) Put y equal to 0.

$$f'(x) = f'(0) + 2x = 2x + c, \qquad c = f'(0).$$

Step 5. (b) Integrate.

$$f(x) = x^2 + cx + d.$$

Using $f(0) = 0$: $d = 0$. So $f(x) = x^2 + cx$.

Step 6. (b) Verify.

$$f(x+y) = (x+y)^2 + c(x+y) = (x^2 + cx) + (y^2 + cy) + 2xy = f(x) + f(y) + 2xy. \checkmark$$

So the solutions are exactly $f(x) = x^2 + cx$, for any constant c.

↪ *Verification matters: the differentiation step only shows that a solution **must** look like this; checking shows every such function **is** a solution.*

Answer. (b) $f(x) = x^2 + cx$ for any constant c

Common errors.

- Forgetting to verify that the functions found actually satisfy the equation.
- Differentiating with respect to x and y at the same time.

How the marks are earned.

M1 **(a)(i)** $x = y = 0$: $f(0) = 2f(0)$
A1 $f(0) = 0$
M1 **(a)(ii)** Expands $g(x + y) = f(x + y) - (x + y)^2$
A1* $= g(x) + g(y)$
M1 **(b)** Differentiates with respect to y: $f'(x + y) = f'(y) + 2x$
A1 $y = 0$: $f'(x) = 2x + c$
M1 Integrates
A1 $f(x) = x^2 + cx + d$
M1 $f(0) = 0 \Rightarrow d = 0$
A1 $f(x) = x^2 + cx$, verified

∎

Question 8 *Exponential functional equations* 10 marks

Find all differentiable functions $h : \mathbb{R} \to \mathbb{R}$, not identically zero, such that

$$h(x+y) = h(x)h(y) \qquad \text{for all real } x, y.$$

You should prove that $h(0) = 1$, that $h(x) > 0$ for all x, and that your answer is complete.

Solution.

The idea.

Special values first: $y = 0$ gives $h(0)$, $y = -x$ shows h is never zero, and $x = y$ shows it is never negative. Then differentiate with respect to y to get the differential equation $h' = kh$, and solve it without dividing by anything that might be zero.

Step 1. Find h(0).

Put $y = 0$: $h(x) = h(x)h(0)$ for all x. Since h is not identically zero, pick x_0 with $h(x_0) \neq 0$ and divide: $h(0) = 1$.

Step 2. h is never zero.

Put $y = -x$: $h(x)h(-x) = h(0) = 1$. A product equal to 1 can't have a zero factor, so $h(x) \neq 0$ for every x.

Step 3. h is positive.

Put $x = y = \frac{t}{2}$:

$$h(t) = h\left(\tfrac{t}{2}\right)^2 \geq 0.$$

Combined with $h(t) \neq 0$: $h(t) > 0$ for all t.

Step 4. A differential equation.

Differentiate $h(x+y) = h(x)h(y)$ with respect to y, then put $y = 0$:

$$h'(x) = h(x)h'(0) = kh(x), \qquad k = h'(0).$$

Step 5. Solve it rigorously.

Consider $u(x) = h(x)e^{-kx}$. By the product rule,

$$u'(x) = h'(x)e^{-kx} - kh(x)e^{-kx} = \big(kh(x) - kh(x)\big)e^{-kx} = 0.$$

So u is constant: $h(x) = Ce^{kx}$, and $C = h(0) = 1$.

↪ *This avoids dividing by h and writing $\int \frac{h'}{h} = \ln h$, which would need $h > 0$. We've proved that, but this route needs nothing.*

Step 6. Verify and conclude.

$e^{k(x+y)} = e^{kx}e^{ky}$ ✓. So the solutions are exactly

$$h(x) = e^{kx}, \qquad k \in \mathbb{R}$$

(including $k = 0$, the constant function 1).

Answer. $h(x) = e^{kx}$ for any real k

Common errors.

- Dividing by $h(x)$ before proving it is never zero.
- Forgetting the constant solution $h(x) = 1$ (the case $k = 0$): it is included in e^{kx}.

Examiner's note.

Without differentiability, (a)(ii) has bizarre non-linear solutions (they need the axiom of choice to construct). That's why the extra condition is there.

How the marks are earned.

M1 $y = 0$: $h(x) = h(x)h(0)$
A1 Uses a point where $h \neq 0$: $h(0) = 1$
M1 $h(x)h(-x) = 1$, so h is never 0
M1 $h(x) = h\left(\frac{x}{2}\right)^2 \geq 0$
A1 Hence $h(x) > 0$
M1 Differentiates with respect to y at $y = 0$: $h'(x) = kh(x)$, $k = h'(0)$
M1 Solves rigorously, e.g. $\frac{d}{dx}\left(h(x)e^{-kx}\right) = 0$
A1 $h(x) = Ce^{kx}$
A1 $C = h(0) = 1$
A1 $h(x) = e^{kx}$ for any real k, verified

∎

Question 9 *The irrationality of π: an integer* 10 marks

In this question you will prove that π is irrational (Niven, 1947). Suppose, for contradiction, that $\pi = \dfrac{a}{b}$, where a and b are positive integers. For a positive integer n, define

$$f(x) = \frac{x^n(a - bx)^n}{n!}$$

and

$$F(x) = f(x) - f''(x) + f^{(4)}(x) - \cdots + (-1)^n f^{(2n)}(x).$$

(a) *Show that $f(\pi - x) = f(x)$, and deduce that $f^{(j)}(\pi) = (-1)^j f^{(j)}(0)$ for every $j \geq 0$.* *(4)*

(b) *Show that $f^{(j)}(0)$ is an integer for every $j \geq 0$. Deduce that $F(0) + F(\pi)$ is an integer.* *(6)*

Solution.

The idea.

If $\pi = \frac{a}{b}$, the polynomial f has a hidden symmetry about $\frac{\pi}{2}$, and all of its derivatives at 0 and π turn out to be whole numbers. That makes $F(0) + F(\pi)$ a whole number, the first ingredient of the contradiction completed in the next question.

Step 1. (a) The symmetry.

Using $a = b\pi$: $a - b(\pi - x) = a - b\pi + bx = bx$, and $a - bx = b(\pi - x)$. So

$$f(\pi - x) = \frac{(\pi - x)^n (bx)^n}{n!}, \qquad f(x) = \frac{x^n\, b^n (\pi - x)^n}{n!}.$$

These are equal. ■

Step 2. (a) Differentiate the symmetry.

Differentiating $f(\pi - x) = f(x)$ once gives $-f'(\pi - x) = f'(x)$ (chain rule). Each further differentiation brings another factor of -1:

$$(-1)^j f^{(j)}(\pi - x) = f^{(j)}(x).$$

Put $x = 0$: $f^{(j)}(\pi) = (-1)^j f^{(j)}(0)$. ■

Step 3. (b) Expand f.

By the binomial theorem, $(a - bx)^n$ is a polynomial with **integer** coefficients (as a, b are integers). Multiplying by x^n:

$$f(x) = \frac{1}{n!} \sum_{m=n}^{2n} c_m x^m, \qquad c_m \in \mathbb{Z}.$$

The lowest power is x^n and the highest is x^{2n}.

Step 4. (b) Derivatives at 0.

Differentiating x^m a total of j times and putting $x = 0$ gives 0 unless $m = j$, in which case it gives $j!$. So

$$f^{(j)}(0) = \frac{j!\, c_j}{n!}.$$

- If $j < n$ or $j > 2n$: there is no x^j term, so $f^{(j)}(0) = 0$.

- If $n \le j \le 2n$: $\frac{j!}{n!} = j(j-1)\cdots(n+1)$ is an integer, so $f^{(j)}(0)$ is an integer.

Step 5. (b) Values at π, and F.

By (a), $f^{(j)}(\pi) = \pm f^{(j)}(0)$, also an integer. $F(0)$ and $F(\pi)$ are sums of such values, so $F(0) + F(\pi) \in \mathbb{Z}$. ■

↪ *This is the only place we used $\pi = \frac{a}{b}$: it made $a - bx$ vanish at $x = \pi$, giving the symmetry in (a).*

Answer. $F(0) + F(\pi)$ is an integer.

Common errors.

- Forgetting that f vanishes to order n at 0, which is what makes the low derivatives zero.
- Missing that $\frac{j!}{n!}$ is an integer only because $j \ge n$.

How the marks are earned.

M1 (a) $a - b(\pi - x) = bx$, using $a = b\pi$
A1 $f(\pi - x) = f(x)$
M1 Differentiates j times: $(-1)^j f^{(j)}(\pi - x) = f^{(j)}(x)$
A1 Puts $x = 0$
M1 (b) $f(x) = \dfrac{1}{n!}\sum_{m=n}^{2n} c_m x^m$ with integer c_m
M1 $f^{(j)}(0) = \dfrac{j!\,c_j}{n!}$
B1 $\dfrac{j!}{n!}$ is an integer for $j \ge n$
A1 Zero for $j < n$ and $j > 2n$; so always an integer
M1 Uses (a) for the values at π
A1 $F(0) + F(\pi)$ is an integer

∎

Question 10 *The irrationality of π: the contradiction* 10 marks

As in the previous question, suppose that $\pi = \dfrac{a}{b}$, where a and b are positive integers, and for a positive integer n let

$$f(x) = \frac{x^n(a-bx)^n}{n!}, \qquad F(x) = f(x) - f''(x) + f^{(4)}(x) - \cdots + (-1)^n f^{(2n)}(x).$$

You may assume that $F(0) + F(\pi)$ is an integer.

(a) *Show that $F''(x) + F(x) = f(x)$, and hence that* *(4)*

$$\int_0^\pi f(x)\sin x\,dx = F(\pi) + F(0).$$

(b) *Show that* *(6)*

$$0 < \int_0^\pi f(x)\sin x\,dx < \frac{\pi^{n+1}a^n}{n!},$$

and hence complete the proof that π is irrational.

A video explanation of this question is at lemmamathematics.com/papers/pure-ultra#q10.

Solution.

The idea.

*The same strategy as for e, but with a cleverer integer. We build an integral that (i) is an **integer** if $\pi = \frac{a}{b}$, (ii) is **positive**, and (iii) can be made **smaller than 1** by taking n large. Nothing can be all three. The video shows step (iii): however big a is, $n!$ eventually wins.*

Step 1. (a) F double dash plus F.

$$F'' = f'' - f^{(4)} + f^{(6)} - \cdots + (-1)^n f^{(2n+2)}.$$

f is a polynomial of degree $2n$, so $f^{(2n+2)} = 0$. Adding F, everything cancels in pairs except the first term of F:

$$F'' + F = f.$$

Step 2. (a) A perfect derivative.

$$\frac{d}{dx}\big(F'(x)\sin x - F(x)\cos x\big) = F''\sin x + F'\cos x - F'\cos x + F\sin x = (F''+F)\sin x = f(x)\sin x.$$

Step 3. (a) Evaluate.

$$\int_0^\pi f(x)\sin x\,dx = \Big[F'(x)\sin x - F(x)\cos x\Big]_0^\pi = \big(0 + F(\pi)\big) - \big(0 - F(0)\big) = F(\pi) + F(0). \quad \blacksquare$$

So **the integral is an integer**.

Step 4. (b) It is positive.

On $(0,\pi)$: $x > 0$, $a - bx = b(\pi - x) > 0$, so $f(x) > 0$; and $\sin x > 0$. A continuous positive integrand gives a positive integral.

Step 5. (b) It is small.

On $[0,\pi]$: $0 \le x \le \pi$ and $0 \le a - bx \le a$, and $0 \le \sin x \le 1$. So

$$f(x)\sin x \le \frac{\pi^n a^n}{n!} \quad\Longrightarrow\quad \int_0^\pi f(x)\sin x\,dx < \pi\cdot\frac{\pi^n a^n}{n!} = \frac{\pi^{n+1}a^n}{n!}$$

(strict, since the bound isn't attained everywhere).

Step 6. (b) Factorials beat powers.

Let $c = \pi a$, so the bound is $\pi\cdot\frac{c^n}{n!}$. For $m \ge 2c$, each extra factor $\frac{c}{m} \le \frac{1}{2}$. So once n passes $2c$, $\frac{c^n}{n!}$ at least halves with every step, and tends to 0. In particular, there is an n with

$$\frac{\pi^{n+1}a^n}{n!} < 1.$$

(The video plots this bound for $a = 22$: it climbs to about 10^{29} before collapsing below 1 at $n = 186$.)

Step 7. (b) The contradiction.

For that n, the integral is an integer (part (a)), and $0 < \text{integral} < 1$. No such integer exists. So the assumption $\pi = \frac{a}{b}$ is false: π is irrational. ■

> **Answer.** π is irrational.

Common errors.

- Forgetting that f vanishes to order n at 0, which is what makes the low derivatives zero.
- Claiming $\frac{c^n}{n!} \to 0$ without any justification.

Examiner's note.

Notice the shape of all three proofs on this paper and the 5★ one: **integer + positive + less than 1 = impossible**. It is one of the most powerful patterns in number theory.

How the marks are earned.

M1 (a) $F'' = f'' - f^{(4)} + \cdots$, with $f^{(2n+2)} = 0$ (degree $2n$)
A1 $F'' + F = f$
M1 $\dfrac{d}{dx}\left(F'\sin x - F\cos x\right) = (F'' + F)\sin x$
A1* Evaluates between 0 and π: $F(\pi) + F(0)$
M1 (b) $f(x)\sin x > 0$ on $(0,\pi)$, so the integral is positive
M1 $x \le \pi$ and $0 \le a - bx \le a$ on $[0,\pi]$
A1 Integral $< \dfrac{\pi^{n+1}a^n}{n!}$
M1 $\dfrac{c^n}{n!} \to 0$, with a reason (e.g. factors $\frac{c}{m} \le \frac{1}{2}$ once $m \ge 2c$)
A1 Chooses n so that the bound is less than 1
A1 Integer strictly between 0 and 1: π is irrational

∎